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Almost Hermitian 6-manifolds revisited

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Abstract

A theorem of Kirichenko states that the torsion 3-form of the characteristic connection of a nearly Kähler manifold is parallel. On the other side, any almost Hermitian manifold of type G_1 admits a unique connection with totally skew-symmetric torsion. In dimension 6, we generalize Kirichenko's theorem and we describe almost Hermitian G_1-manifolds with parallel torsion form. In particular, among them there are only two types of $\mathcal{W}_3$-manifolds with a non-Abelian holonomy group, namely twistor spaces of four-dimensional self-dual Einstein manifolds and the invariant Hermitian structure on the Lie group $\mathrm{SL}(2, \mathbb{C})$. Moreover, we classify all naturally reductive Hermitian $\mathcal{W}_3$-manifolds with small isotropy group of the characteristic torsion.

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1. Introduction

Fix a subgroup $\mathrm{G} \subset \mathrm{SO}(n)$ of the special orthogonal group and decompose the Lie algebra $\mathfrak{so}(n) = \mathfrak{g} \oplus \mathfrak{m}$ into the Lie algebra $\mathfrak{g}$ of G and its orthogonal complement $\mathfrak{m}$. The different geometric types of G-structures on a Riemannian manifold correspond to the irreducible G-components of the representation $\mathbb{R}^n \otimes \mathfrak{m}$. This approach to non-integrable geometries is a kind of folklore in differential geometry, and was exposed in detail in the article [19]. Indeed, consider a Riemannian manifold (M^n, g) and denote its Riemannian

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frame bundle by $\mathcal{F}(M^n)$. It is a principal $\mathrm{SO}(n)$-bundle over M^n. A G-structure is a reduction $\mathcal{R} \subset \mathcal{F}(M^n)$ of the frame bundle to the subgroup G. The Levi–Civita connection is a 1-form Z on $\mathcal{F}(M^n)$ with values in the Lie algebra $\mathfrak{so}(n)$. We restrict the Levi–Civita connection to $\mathcal{R}$ and decompose it with respect to the decomposition of the Lie algebra $\mathfrak{so}(n)$,

$$Z|_{T(\mathcal{R})} := Z^* \oplus \Gamma.$$

Then, Z^* is a connection in the principal G-bundle $\mathcal{R}$ and Γ is a 1-form on M^n with values in the associated bundle $\mathcal{R} \times_{\mathrm{G}} \mathfrak{m}$. Suppose that the group G and the G-structure are defined by some differential form T. Examples are almost Hermitian structures or almost metric contact structures. Then the Riemannian covariant derivative of T is given by the formula

$$\nabla^{\mathrm{LC}}\mathrm{T} = \varrho_*(\Gamma)(\mathrm{T}),$$

where $\varrho_*(\Gamma)(\mathrm{T})$ denotes the algebraic action of the 2-form Γ on T. Some authors call Γ the *intrinsic torsion* of the G-structure. There is a second notion, namely the *characteristic connection* and the *characteristic torsion* of a G-structure. It is a G-connection ∇^{c} with totally skew-symmetric torsion tensor. Not any type of geometric G-structures admits a characteristic connection. In order to formulate the condition, we embed the space of all 3-forms into $\mathbb{R}^n \otimes \mathfrak{m}$ using the morphism

$$\Theta : \Lambda^3(\mathbb{R}^n) \to \mathbb{R}^n \otimes \mathfrak{m}, \qquad \Theta(\mathrm{T}) := \sum_{i=1}^{n} e_i \otimes \mathrm{pr}_{\mathfrak{m}}(e_i \lrcorner \mathrm{T}).$$

A geometric G-structure admits a characteristic connection ∇^{c} if and only if the intrinsic torsion Γ belongs to the image of the Θ. In this case, the intrinsic torsion is given by the equation (see [19,20])

$$2\Gamma = -\Theta(\mathrm{T}^{\mathrm{c}}).$$

For several geometric structures the characteristic torsion form has been computed explicitly in terms of the underlying geometric data. Formulas of that type are known for almost Hermitian structures, almost metric contact structures and G_2-structures in dimension 7 (see [22]). For a Riemannian naturally reductive space $M^n = \mathrm{G}_1/\mathrm{G}$, we obtain a G-reduction $\mathcal{R} := \mathrm{G}_1 \subset \mathcal{F}(M^n)$ of the frame bundle. Then the characteristic connection of the G-structure coincides with the *canonical* connection of the reductive space. In this sense, we can understand the characteristic connection of a Riemannian G-structure as a generalization of the canonical connection of a Riemannian naturally reductive space. The canonical connection of a naturally reductive space has parallel torsion form and parallel curvature tensor

$$\nabla^{\mathrm{c}}\mathrm{T}^{\mathrm{c}} = 0, \qquad \nabla^{\mathrm{c}}\mathrm{R}^{\mathrm{c}} = 0.$$

For arbitrary G-structures and their characteristic connections, these properties do not hold anymore. Corresponding examples are discussed in [22]. However, the parallelism of the torsion form is an important property. The first reason is that $\nabla^{\mathrm{c}}\mathrm{T}^{\mathrm{c}} = 0$ implies the conservation law $\delta(\mathrm{T}^{\mathrm{c}}) = 0$, one of the conditions for the NS-3-form in type II string theory (for constant dilaton). Moreover, if the torsion is parallel, several formulas for differential operators acting on spinors simplify (see [5]) and it is possible to investigate—via integral formulas—the space of parallel or harmonic spinors. Sasakian structures or nearly Kähler structures have a parallel characteristic torsion form, even if they are not reductive. This motivates the investigation of Riemannian G-structures with a parallel characteristic torsion form in general. In this paper, we study the problem for almost Hermitian manifolds in dimension 6.

First we revisit almost Hermitian manifolds in real dimension 6. The Hodge operator acts as a complex structure on $\Lambda^3(\mathbb{R}^6)$. This observation simplifies, in dimension 6, the description of the algebraic decomposition of the space of all 3-forms $\Lambda^3(\mathbb{R}^6)$ and of the space $\mathbb{R}^6 \otimes \mathfrak{m}^6$ containing the intrinsic torsion. We develop the algebraic part needed for the classification of almost Hermitian structures and we compute the corresponding differential equations characterizing the 16 classes of almost Hermitian manifolds (see [13,15,29]). It is a basic property of six-dimensional nearly Kähler manifolds that their characteristic torsion T^{c} is ∇^{c}-parallel. The necessary formulas proving that fact have been derived by the Japanese school at the beginning of the 70-ties of the last century (see [35,38,40]). Later Gray [27,28] and Kirichenko [33] used these curvature identities for the investigation of the geometry of nearly Kähler manifolds. However, the ∇^{c}-parallelism of the characteristic torsion T^{c} was explicitly formulated only recently (see [10,22,33]). We outline a short proof here, and continue our investigation along this path. Any almost Hermitian manifold of type G_1 admits a unique characteristic connection (see [22]). We study almost Hermitian G_1-manifolds with a parallel characteristic torsion. The $\mathrm{U}(3)$-orbit type of the characteristic torsion is constant. There are two possibilities. If the vector part of the intrinsic torsion is non-trivial, we obtain two commuting Killing vector fields of constant length, and the manifold is a torus fibration over some special 4-manifold. If the vector part vanishes, we list the relevant $\mathrm{U}(3)$-orbit types of the torsion 3-forms. It turns out that there exist only two orbits with a non-Abelian isotropy (holonomy) group in dimension 6. These two types can be realized and the corresponding Hermitian manifolds are twistor spaces or the invariant, non-Kählerian Hermitian structure on the Lie group $\mathrm{SL}(2,\mathbb{C})$. Finally we classify all naturally reductive Hermitian $\mathcal{W}_3$-manifolds with small isotropy group of the characteristic torsion.

2. Almost complex structures in real dimension 4

2.1. The subgroup $\mathrm{U}(k)$ in $\mathrm{SO}(2k)$

We start with some notations that will be used throughout this paper. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. Using its scalar product $\langle\, ,\, \rangle$, we identify Euclidean space with its dual space, $\mathbb{R}^n = (\mathbb{R}^n)^*$. $e_1, \ldots, e_n$ is an orthonormal basis in $\mathbb{R}^n$. $\Lambda^l(\mathbb{R}^n)$ is the space of l-forms in $\mathbb{R}^n$. $e_{i_1,\ldots,i_l}$ means the exterior product $e_{i_1} \wedge \cdots \wedge e_{i_l}$ of the corresponding 1-forms. We decompose a 2-form ω or a 3-form T into their components,

$$\omega = \sum_{1\le i<j\le n} w_{ij} \cdot e_{ij}, \qquad \mathrm{T} = \sum_{1\le i<j<k\le n} \mathrm{T}_{ijk} \cdot e_{ijk}.$$

The special orthogonal group $\mathrm{SO}(n)$ acts on $\Lambda^l(\mathbb{R}^n)$ and the differential $\varrho_* : \mathfrak{so}(n) \to \mathrm{End}(\Lambda^l(\mathbb{R}^n))$ of this representation is given by

$$\varrho_*(\omega)(\mathrm{T}) = \sum_{i=1}^{n} (e_i \lrcorner\, \omega) \wedge (e_i \lrcorner\, \mathrm{T}).$$

The space of 2-forms $\Lambda^2(\mathbb{R}^n) = \mathfrak{so}(n)$ coincides with the Lie algebra of the special orthogonal group, ϱ is the adjoint representation and its differential ϱ_* coincides with the commutator action.

We consider the complex structure $\mathrm{J} : \mathbb{R}^{2k} \to \mathbb{R}^{2k}$ of the even-dimensional Euclidean space. With respect to the standard orthonormal basis it is given by

$$\mathrm{J}e_{2i-1} = e_{2i}, \qquad \mathrm{J}e_{2i} = -e_{2i-1}, \quad i = 1, 2, \dots, k.$$

The subgroup $\mathrm{U}(k) \subset \mathrm{SO}(2k)$ consists of all orthogonal transformations commuting with the complex structure

$$\mathrm{U}(k) := \{A \in \mathrm{SO}(2k) : A \circ \mathrm{J} = \mathrm{J} \circ A\}.$$

The Lie algebra $\mathfrak{so}(2k)$ splits into the subalgebra $\mathfrak{u}(k)$ and its orthogonal complement $\mathfrak{m}$,

$$\mathfrak{so}(2k) = \Lambda^2(\mathbb{R}^{2k}) = \mathfrak{u}(k) \oplus \mathfrak{m}.$$

The complex structure J acts on $\Lambda^2(\mathbb{R}^{2k})$ as an involution. Using this involution, we can describe the spaces of the decomposition

$$\mathfrak{u}(k) = \{\omega \in \Lambda^2(\mathbb{R}^{2k}) : \mathrm{J}(\omega) = \omega\}, \qquad \mathfrak{m} = \{\omega \in \Lambda^2(\mathbb{R}^{2k}) : \mathrm{J}(\omega) = -\omega\}.$$

The center of the Lie algebra $\mathfrak{u}(k)$ is generated by the 2-form $\Omega(X, Y) := g(\mathrm{J}(X), Y)$ and the Lie algebra splits into

$$\mathfrak{u}(k) = \mathfrak{su}(k) \oplus \mathbb{R}^1 \cdot \Omega.$$

The Lie algebra $\mathfrak{u}(k)$ is the space of all 2-forms defined by the equations

$$w_{2i-1,2j-1} - w_{2i,2j} = 0, \qquad -w_{2i-1,2j} + w_{2i,2j-1} = 0, \quad 1 \le i < j \le k.$$

The additional equation singling out the Lie algebra $\mathfrak{su}(k)$ inside $\mathfrak{u}(k)$ is

$$w_{12} + w_{34} + \cdots + w_{2k-1,2k} = 0.$$

2.2. *The decomposition of* $\mathbb{R}^4 \otimes \mathfrak{m}^2$

In dimension 4, the Hodge operator as well as the complex structure act on 2-forms as involutions

$$\mathrm{J}^2 = \mathrm{Id} = *^2, \qquad \mathrm{J} \circ * = * \circ \mathrm{J}.$$

In contrast to the higher-dimensional case, in real dimension 4 there are only two types. They are determined by the Nijenhuis tensor and the differential of the Kähler form. In order to understand the geometric types of $\mathrm{U}(2)$-structures on four-dimensional Riemannian manifolds, we need the decomposition of the representation $\mathbb{R}^4 \otimes \mathfrak{m}^2$. Denote by

$$\Phi : \mathbb{R}^4 \otimes \mathfrak{m}^2 \to \Lambda^3(\mathbb{R}^4), \qquad \Phi(X \otimes \omega^2) := X \wedge \omega^2,$$

the total anti-symmetrization of a tensor in $\mathbb{R}^4 \otimes \mathfrak{m}^2$. On the other side, we embed the space of all 3-forms into $\mathbb{R}^4 \otimes \mathfrak{m}^2$ using the morphism $\Theta : \Lambda^3(\mathbb{R}^4) \to \mathbb{R}^4 \otimes \mathfrak{m}^2$ defined in Section 1. A direct algebraic computation proves the following Lemma.

Lemma 2.1. *The morphism* $\Phi : \mathbb{R}^4 \otimes \mathfrak{m}^2 \to \Lambda^3(\mathbb{R}^4)$ *is surjective and* $\Phi \circ \Theta$ *acts on the space of all* 3*-forms by*

$$\Phi \circ \Theta = \mathrm{Id}.$$

Let us introduce two $\mathrm{U}(2)$-invariant subspaces of $\mathbb{R}^4 \otimes \mathfrak{m}^2$,

$$\mathcal{W}_2 := \mathrm{Ker}(\Phi), \qquad \mathcal{W}_4 := \Theta(\Lambda^3(\mathbb{R}^4)).$$

Obviously, $\mathbb{R}^4 \otimes \mathfrak{m}^2$ splits under the action of the group $\mathrm{U}(2)$ into these subspaces.

Proposition 2.1. $\mathcal{W}_2$ *and* $\mathcal{W}_4$ *are real, irreducible* $\mathrm{U}(2)$*-representations*.

Proof. We restrict the representation $\mathbb{R}^4 \otimes \mathfrak{m}^2$ to the subgroup $\mathrm{SU}(2)$. Then $\mathfrak{m}^2$ is trivial and $\mathbb{R}^4 \otimes \mathfrak{m}^2 = \mathbb{R}^4 \oplus \mathbb{R}^4$ splits into two irreducible components under the action of $\mathrm{SU}(2)$. □

2.3. Geometric types of almost Hermitian 4-manifolds

Consider an almost Hermitian manifold (M^4, g, J) and denote its Riemannian frame bundle by $\mathcal{F}(M^4)$. The almost Hermitian structure is a reduction $\mathcal{R} \subset \mathcal{F}(M^4)$ of the frame bundle to the subgroup $\mathrm{U}(2)$. The different non-integrable types of Hermitian structures are the irreducible components of the representation $\mathbb{R}^4 \otimes \mathfrak{m}^2$. We split the intrinsic torsion Γ,

$$\Gamma = \Gamma_4 \oplus \Gamma_4^*.$$

Note that, via the identification Θ, Γ_4 is an ordinary 3-form on the Hermitian manifold. Moreover, in real dimension 4, the differential and the co-differential of the Kähler form coincide,

$$\delta\Omega = - * d * \Omega = - * d\Omega.$$

The co-differential of the Kähler form is given by the formula

$$-\delta\Omega = \sum_{i=1}^{4} e_i \lrcorner \nabla^{\mathrm{LC}}_{e_i}\Omega = \sum_{i=1}^{4} \{\Gamma(e_i)(e_i \lrcorner \Omega, -) - \Omega(e_i \lrcorner \Gamma(e_i), -)\} =: \Pi(\Gamma).$$

The map $\Pi : \mathbb{R}^4 \otimes \mathfrak{m}^2 \to \mathbb{R}^4$ is obviously $\mathrm{U}(2)$-equivariant. Consequently, the co-differential $\delta\Omega$ depends only on the Γ_4-part of the intrinsic torsion.

Proposition 2.2. *Let $\Gamma_4 = \Theta(\mathrm{T}_4)$ be given by the 3-form $\mathrm{T}_4 \in \Lambda^3(\mathbb{R}^4)$. Then $\Pi(\Gamma_4) = -2\mathrm{J}(*\mathrm{T}_4)$ holds. In particular, the co-differential of the Kähler form of any almost Hermitian 4-manifold is given by the formula*

$$- * d\Omega = \delta\Omega = 2\mathrm{J}(*\mathrm{T}_4).$$

The Nijenhuis tensor in real dimension 4 has four components. Setting $\mathrm{N}_1 := \mathrm{N}(e_1, e_3)$, it may be written in the form

$$\mathrm{N} = (e_{13} - e_{24}) \otimes \mathrm{N}_1 - (e_{23} + e_{14}) \otimes \mathrm{J}(\mathrm{N}_1).$$

The anti-symmetrization map $\Phi : \mathbb{R}^4 \otimes \mathfrak{m}^2 \to \Lambda^3(\mathbb{R}^4)$ vanishes on the Nijenhuis tensor, i.e., N is an element of the subspace $\mathcal{W}_2$. Consequently, there are two basic geometric types of almost Hermitian 4-manifolds. They correspond to the Nijenhuis tensor (the Γ_4^*-part) and to the differential $d\Omega$ of the Kähler form (the Γ_4-part). An almost Hermitian 4-manifold admits a characteristic connection if and only if its Nijenhuis tensor vanishes (Hermitian manifold). In this case, the characteristic torsion is given by the formula $\mathrm{T}^c = -\mathrm{J}(d\Omega)$. It is ∇^c-parallel if and only if the Lee form $\delta\Omega \circ \mathrm{J}$ is parallel with respect to the Levi–Civita connection. Hermitian manifolds of that type are called generalized Hopf manifolds (see [39]). The compact four-dimensional generalized Hopf manifolds have been described by Belgun [9].

3. Almost complex structures in real dimension 6

In real dimension 6, the Hodge operator as well as the complex structure act on 3-forms as complex structures. Moreover, the central element $\Omega \in \mathfrak{u}(3)$ acts on 3-forms, too. These three operators split the spaces $\Lambda^3(\mathbb{R}^6)$ and $\mathbb{R}^6 \otimes \mathfrak{m}^6$ into U(3)-irreducible components. There are four basic types of almost Hermitian 6-manifolds. They are characterized by the components of the derivative $d\Omega$ and the Nijenhuis tensor N.

3.1. The decomposition of $\Lambda^3(\mathbb{R}^6)$

Two operators act on the space $\Lambda^3(\mathbb{R}^6)$, namely J and the Hodge operator $*$. The complex structure acts on a 3-form T by

$$(\mathrm{JT})(X, Y, Z) := \mathrm{T}(\mathrm{J}X, \mathrm{J}Y, \mathrm{J}Z).$$

We obtain a $(\mathbb{Z}_4 \oplus \mathbb{Z}_4)$-action on the space of all 3-forms

$$\mathrm{J}^2 = -\mathrm{Id}, \quad *^2 = -\mathrm{Id}, \qquad \mathrm{J} \circ * = * \circ \mathrm{J}.$$

Let us decompose $\Lambda^3(\mathbb{R}^6)$ into two U(3)-invariant subspaces

$$\Lambda^3(\mathbb{R}^6) := \Lambda^3_+(\mathbb{R}^6) \oplus \Lambda^3_-(\mathbb{R}^6), \quad \Lambda^3_\pm := \{\mathrm{T} \in \Lambda^3(\mathbb{R}^6) : \mathrm{J}(\mathrm{T}) = \pm * \mathrm{T}\}.$$

We embed the standard representation $\mathbb{R}^6$ into the 3-forms

$$\Lambda^3_6(\mathbb{R}^6) := \{X \wedge \Omega : X \in \mathbb{R}^6\}.$$

Lemma 3.1. *The spaces* $\Lambda^3_-(\mathbb{R}^6) = \Lambda^3_6(\mathbb{R}^6)$ *coincide. The space* $\Lambda^3_+(\mathbb{R}^6)$ *is its orthogonal complement*

$$\Lambda^3_+(\mathbb{R}^6) = \{\mathrm{T} : \Omega \wedge \mathrm{T} = 0\} = \{\mathrm{T} : \Omega \wedge *\mathrm{T} = 0\} = \{\mathrm{T} : \mathrm{J}(\mathrm{T}) = *\mathrm{T}\}.$$

The $\mathrm{U}(3)$-representation $\Lambda^3_+(\mathbb{R}^6)$ is not irreducible. In order to decompose it, we consider the action of the central element $\Omega \in \mathfrak{u}(3)$ on the space $\Lambda^3(\mathbb{R}^6)$. We will denote by $\tau : \Lambda^3(\mathbb{R}^6) \to \Lambda^3(\mathbb{R}^6)$ this special anti-symmetric operator acting on 3-forms

$$\tau(\mathrm{T}) := \varrho_*(\Omega)(\mathrm{T}) = \sum_{i=1}^{6} (e_i \lrcorner\, \Omega) \wedge (e_i \lrcorner\, \mathrm{T}).$$

Lemma 3.2. *The symmetric endomorphism* τ^2 *has two eigenvalues and splits the space* $\Lambda^3_+(\mathbb{R}^3)$ *into a two-dimensional and a* 12*-dimensional space*

$$\Lambda^3_+(\mathbb{R}^6) := \Lambda^3_2(\mathbb{R}^6) \oplus \Lambda^3_{12}(\mathbb{R}^6),$$

where

$$\Lambda^3_2(\mathbb{R}^6) := \{\mathrm{T} \in \Lambda^3(\mathbb{R}^6) : \tau^2(\mathrm{T}) = -9\mathrm{T}, \mathrm{J}(\mathrm{T}) = *\mathrm{T}\},$$
$$\Lambda^3_{12}(\mathbb{R}^6) := \{\mathrm{T} \in \Lambda^3(\mathbb{R}^6) : \tau^2(\mathrm{T}) = -\mathrm{T}, \mathrm{J}(\mathrm{T}) = *\mathrm{T}\}.$$

The anti-symmetric endomorphism τ *preserves any of these spaces and acts as*

$$\tau(\mathrm{T}) = 3 * \mathrm{T} \ \ \textit{on}\ \Lambda^3_2(\mathbb{R}^6), \qquad \tau(\mathrm{T}) = *\mathrm{T} \ \ \textit{on}\ \Lambda^3_6(\mathbb{R}^6), \qquad \tau(\mathrm{T}) = - * \mathrm{T} \ \ \textit{on}\ \Lambda^3_{12}(\mathbb{R}^6).$$

It is useful to have at hand an explicit basis in any of these spaces:

in $\Lambda^3_2(\mathbb{R}^6)$: $\quad -e_{246} + e_{136} + e_{145} + e_{235}, \qquad -e_{135} + e_{245} + e_{236} + e_{146},$

in $\Lambda^3_6(\mathbb{R}^6)$: $\quad e_{134} + e_{156}, \quad e_{234} + e_{256}, \quad e_{123} + e_{356}, \quad e_{124} + e_{456},$
$e_{125} + e_{345}, \quad e_{126} + e_{346},$

in $\Lambda^3_{12}(\mathbb{R}^6)$: $\quad e_{123} - e_{356}, \quad e_{124} - e_{456}, \quad e_{125} - e_{345}, \quad e_{126} - e_{346},$
$e_{134} - e_{156}, \quad e_{234} - e_{256}, \quad e_{135} + e_{245}, \quad e_{246} + e_{136},$
$e_{135} + e_{236}, \quad e_{246} + e_{145}, \quad e_{135} + e_{146}, \quad e_{246} + e_{235}.$

Any two-dimensional real representation of the simply connected compact Lie group $\mathrm{SU}(3)$ is trivial. Therefore, the subgroup $\mathrm{SU}(3)$ preserves any 3-form in $\Lambda^3_2(\mathbb{R}^6)$ and we can understand the subgroup $\mathrm{SU}(3) \subset \mathrm{U}(3)$ as the isotropy group of a 3-form of that type. Moreover, we obtain an $\mathrm{SU}(3)$-isomorphism between $\mathbb{R}^6$ and $\mathfrak{m}^6$.

Corollary 3.1. *For any non-trivial* 3*-form* $\mathrm{T} \in \Lambda^3_2(\mathbb{R}^6)$, *the map* $X \to X \lrcorner\, \mathrm{T}$ *defines an* $\mathrm{SU}(3)$*-isomorphism between* $\mathbb{R}^6$ *and* $\mathfrak{m}^6$.

Remark 3.1. The irreducible six-dimensional $\mathrm{U}(3)$-representation $\mathfrak{m}^6$ is not equivalent to the standard representation in $\mathbb{R}^6$. Indeed, J is an element of the group $\mathrm{U}(3)$ and we compute its trace in $\mathbb{R}^6$ and in $\mathfrak{m}^6$, $\mathrm{Tr}_{\mathbb{R}^6}(\mathrm{J}) = 0$, $\mathrm{Tr}_{\mathfrak{m}^6}(\mathrm{J}) = -6$.

Now we prove that $\Lambda^3_{12}(\mathbb{R}^6)$ is irreducible. We use the fact that there exists only one non-trivial six-dimensional real representation of the group $\mathrm{SU}(3)$. For completeness, we sketch the proof, too.

Proposition 3.1. *Any six-dimensional real representation of the group* $\mathrm{SU}(3)$ *is either trivial or isomorphic to the standard representation in* $\mathbb{R}^6 = \mathbb{C}^3$.

Proof. The Euclidean group $\mathrm{SO}(5)$ does not contain a subgroup of dimension 8. Consequently, any six-dimensional real representation V^6 of $\mathrm{SU}(3)$ is either trivial or irreducible. Suppose that $V^6 \neq \mathbb{R}^6$ is irreducible. Since $\mathbb{C}^3$ and its complex conjugation are the only complex irreducible representation of the group $\mathrm{SU}(3)$ in dimension 3, the complexification $(V^6)^{\mathbb{C}}$ must be irreducible (see [2, Lemma 3.58]). Again, there are only two six-dimensional irreducible $\mathrm{SU}(3)$-representations, namely $\mathrm{Sym}^2(\mathbb{C}^3)$ and its conjugation. We compute the character of the element $g = \mathrm{diag}(z, z, z^{-2}) \in \mathrm{SU}(3)$,

$$\chi_{\mathrm{Sym}^2(\mathbb{C}^3)}(g) = 3z^2 + z^{-4} + 2z^{-1},$$

and conclude that $\mathrm{Sym}^2(\mathbb{C}^3)$ is not a real representation. □

Theorem 3.1. *The decomposition*

$$\Lambda^3(\mathbb{R}^6) = \Lambda^3_2(\mathbb{R}^6) \oplus \Lambda^3_6(\mathbb{R}^6) \oplus \Lambda^3_{12}(\mathbb{R}^6)$$

splits the space of all 3*-forms into irreducible, real* $\mathrm{U}(3)$*-representations. Moreover,* $\Lambda^3_6(\mathbb{R}^6)$ *and* $\Lambda^3_{12}(\mathbb{R}^6)$ *are irreducible* $\mathrm{SU}(3)$*-representations.* $\Lambda^3_2(\mathbb{R}^6)$ *is the trivial two-dimensional real* $\mathrm{SU}(3)$*-representation.* $\Lambda^3_2(\mathbb{R}^6)$, $\Lambda^3_6(\mathbb{R}^6)$ *and* $\Lambda^3_{12}(\mathbb{R}^6)$ *are irreducible, complex representations of dimensions* 1, 3 *and* 6, *respectively.*

Proof. The $\mathrm{SU}(3)$-representation $\Lambda^3_{12}(\mathbb{R}^6)$ can split only into $\mathbb{R}^6 \oplus \mathbb{R}^6$ or $\mathbb{R}^6 \oplus 6\mathbb{R}^1$ (see Proposition 3.1). Consider the following elements of the group $\mathrm{SU}(3)$:

$$g_1 := \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad g_2 := \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

We compute the values of the characters

$$\chi_{\Lambda^3_{12}(\mathbb{R}^6)}(g_1) = 4, \qquad \chi_{\mathbb{R}^6 \oplus \mathbb{R}^6}(g_1) = -4, \qquad \chi_{\Lambda^3_{12}(\mathbb{R}^6)}(g_2) = 0,$$
$$\chi_{\mathbb{R}^6 \oplus 6\mathbb{R}^1}(g_2) = 8,$$

i.e., both cases are impossible. □

3.2. The decomposition of $\mathbb{R}^6 \otimes \mathfrak{m}^6$

In order to understand the geometric types of U(3)-structures on six-dimensional Riemannian manifolds, we need the decomposition of the representation $\mathbb{R}^6 \otimes \mathfrak{m}^6$. Denote by

$$\Phi : \mathbb{R}^6 \otimes \mathfrak{m}^6 \to \Lambda^3(\mathbb{R}^6), \qquad \Phi(X \otimes \omega^2) := X \wedge \omega^2,$$

the total anti-symmetrization of a tensor in $\mathbb{R}^6 \otimes \mathfrak{m}^6$. On the other side, we embed the space of all 3-forms into $\mathbb{R}^6 \otimes \mathfrak{m}^6$ using the morphism

$$\Theta : \Lambda^3(\mathbb{R}^6) \to \mathbb{R}^6 \otimes \mathfrak{m}^6, \qquad \Theta(\mathrm{T}) := \sum_{i=1}^{6} e_i \otimes \mathrm{pr}_{\mathfrak{m}^6}(e_i \lrcorner \mathrm{T}).$$

A direct algebraic computation yields the following Lemma.

Lemma 3.3. *The morphism* $\Phi : \mathbb{R}^6 \otimes \mathfrak{m}^6 \to \Lambda^3(\mathbb{R}^6)$ *is surjective and* $\Phi \circ \Theta$ *acts on the space of all* 3*-forms by*

$$\Phi \circ \Theta = 3\mathrm{Id} \ \textit{on} \ \Lambda^3_2(\mathbb{R}^6), \qquad \Phi \circ \Theta = \mathrm{Id} \ \textit{on} \ \Lambda^3_6(\mathbb{R}^6) \oplus \Lambda^3_{12}(\mathbb{R}^6).$$

Let us introduce four U(3)-invariant subspaces of $\mathbb{R}^6 \otimes \mathfrak{m}^6$,

$$\mathcal{W}_1 := \Theta(\Lambda^3_2(\mathbb{R}^6)), \qquad \mathcal{W}_2 := \mathrm{Ker}(\Phi), \qquad \mathcal{W}_3 := \Theta(\Lambda^3_{12}(\mathbb{R}^6)),$$
$$\mathcal{W}_4 := \Theta(\Lambda^3_6(\mathbb{R}^6)).$$

$\mathbb{R}^6 \otimes \mathfrak{m}^6$ splits under the action of the group U(3) into these subspaces. We investigate the representation $\mathcal{W}_2$. It splits as an SU(3)-representation. Fix a 3-form $\mathrm{T} \in \Lambda^3_2(\mathbb{R}^6)$. The group SU(3) stabilizes T and the morphism

$$\Psi_{\mathrm{T}} : \mathbb{R}^6 \otimes \mathfrak{m}^6 \to \Lambda^2(\mathbb{R}^6), \qquad \Psi_{\mathrm{T}}(X \otimes \omega^2) := *((X \lrcorner \mathrm{T}) \wedge \omega^2))$$

is SU(3)-equivariant. We can control the image of Ψ_{T}.

Lemma 3.4. *For any non-trivial form* $\mathrm{T} \in \Lambda^3_2(\mathbb{R}^6)$*, the image of* Ψ_{T} *is contained in the Lie algebra* $\mathfrak{u}(3)$*. Moreover,* Ψ_{T} *maps* $\mathcal{W}_2$ *surjectively onto the Lie algebra* $\mathfrak{su}(3)$*.*

Now we decompose the representation $\mathcal{W}_2$ under the action of the group SU(3).

Theorem 3.2. *Fix two linearly independent* 3*-forms* $\mathrm{T}_1, \mathrm{T}_2$ *in* $\Lambda^3_2(\mathbb{R}^6)$*. Then the map*

$$\Psi_{\mathrm{T}_1} \oplus \Psi_{\mathrm{T}_2} : \mathcal{W}_2 \to \mathfrak{su}(3) \oplus \mathfrak{su}(3)$$

is an isomorphism of SU(3)*-representations.*

Finally we prove that $\mathcal{W}_2$ is U(3)-irreducible.

Theorem 3.3. $\mathcal{W}_2$ *is a real, irreducible* U(3)*-representation of dimension* 16.

Proof. The element $e_1 \otimes (e_{14}+e_{23})+e_2 \otimes (e_{13}-e_{24}) \in \mathcal{W}_2$ is not invariant under the action of the 1-parameter group generated by the central element $\Omega \in \mathfrak{u}(3)$. Suppose that $\mathcal{W}_2$ is U(3)-reducible. Then the adjoint representation of SU(3) extends to a representation κ of the group U(3). In particular, $\Omega \in \mathfrak{u}(3)$ defines a non-trivial, skew-symmetric SU(3)-invariant operator $\kappa_*(\Omega) : \mathfrak{su}(3) \to \mathfrak{su}(3)$. Since for any simple Lie group G we have

$$\Lambda^2(\mathfrak{g})^{\mathrm{G}} = 0,$$

this is a contradiction. □

Corollary 3.2. *The* U(3)*-representation* $\mathbb{R}^6 \otimes \mathfrak{m}^6$ *splits into four irreducible representations*

$$\mathbb{R}^6 \otimes \mathfrak{m}^6 = \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4.$$

The SU(3)*-representation* $\mathbb{R}^6 \otimes \mathfrak{m}^6$ *splits into*

$$\mathbb{R}^6 \otimes \mathfrak{m}^6 = \mathbb{R}^2 \oplus (\mathfrak{su}(3) \oplus \mathfrak{su}(3)) \oplus \mathcal{W}_3 \oplus \mathcal{W}_4.$$

3.3. The 16 classes of almost Hermitian structures

Consider an almost Hermitian manifold (M^6, g, J) and denote its Riemannian frame bundle by $\mathcal{F}(M^6)$. The almost Hermitian structure is a reduction $\mathcal{R} \subset \mathcal{F}(M^6)$ of the frame bundle to the subgroup U(3). We restrict the Levi–Civita connection to $\mathcal{R}$ and decompose it with respect to the decomposition of the Lie algebra $\mathfrak{so}(6)$:

$$Z|_{T(\mathcal{R})} := Z^* \oplus \Gamma.$$

The Riemannian covariant derivative of the Kähler form is given by the formula

$$(\nabla^{\mathrm{LC}}_X \Omega)(Y, Z) = \Gamma(X)(Y \lrcorner\, \Omega, Z) - \Omega(Y \lrcorner\, \Gamma(X), Z).$$

The basic types of Hermitian structures are the irreducible components of the representation $\mathbb{R}^6 \otimes \mathfrak{m}^6$. We split the intrinsic torsion Γ,

$$\Gamma = \Gamma_2 \oplus \Gamma_6 \oplus \Gamma_{12} \oplus \Gamma_{16}.$$

Note that, via the identification Θ, Γ_2 and Γ_{12} are 3-forms on the Hermitian manifold and $\Gamma_6 = \Theta(X \wedge \Omega)$ is a vector field.

3.4. The co-differential $\delta\Omega$

The co-differential of any exterior form α on a Riemannian manifold is given by the formula

$$\delta\alpha = -\sum_{i=1}^{n} e_i \lrcorner\, \nabla^{\mathrm{LC}}_{e_i} \alpha.$$

Inserting the formula for the covariant derivative of the Kähler form, we obtain

$$-\delta\Omega = \sum_{i=1}^{6}\{\Gamma(e_i)(e_i \lrcorner \Omega, -) - \Omega(e_i \lrcorner \Gamma(e_i), -)\} =: \Pi(\Gamma).$$

The map $\Pi : \mathbb{R}^6 \otimes \mathfrak{m}^6 \to \mathbb{R}^6$ is obviously U(3)-equivariant. Consequently, the co-differential $\delta\Omega$ depends only on the $\mathcal{W}_4$-part of the intrinsic torsion. We compute the relation explicitly.

Proposition 3.2. *Let $\Gamma_6 = \Theta(X \wedge \Omega)$ be given by the vector $X \in \mathbb{R}^6$. Then $\Pi(\Gamma_6) = 4X$ holds. In particular, the co-differential of the Kähler form of any almost Hermitian manifold is given by the formula*

$$\delta\Omega = -4X, \quad \Gamma_6 = \Theta(X \wedge \Omega).$$

3.5. The differential $d\Omega$

We handle the differential of the Kähler form in a similar way. Indeed, the differential of an arbitrary exterior form on a Riemannian manifold can be computed by the formula

$$d\alpha = \sum_{i=1}^{n} e_i \wedge \nabla^{\mathrm{LC}}_{e_i}\alpha.$$

Inserting again the formula for the covariant derivative of the Kähler form, we obtain

$$d\Omega = \frac{1}{2}\sum_{i,j=1}^{6} e_i \wedge e_j \wedge \{\Gamma(e_i)(e_j \lrcorner \Omega, -) - \Omega(e_j \lrcorner \Gamma(e_i), -)\} =: \Pi_1(\Gamma).$$

The map $\Pi_1 : \mathbb{R}^6 \otimes \mathfrak{m}^6 \to \Lambda^3(\mathbb{R}^6)$ is obviously U(3)-equivariant. Consequently, the differential $\mathrm{d}\Omega$ depends only on the $\Theta(\Lambda^3(\mathbb{R}^6))$-part of the intrinsic torsion. Moreover, we need a formula for the endomorphism $\Pi_1 \circ \Theta : \Lambda^3(\mathbb{R}^6) \to \Lambda^3(\mathbb{R}^6)$.

Proposition 3.3. *The endomorphism $\Pi_1 \circ \Theta$ is given on the irreducible components by the formulas*

(1) $\Pi_1 \circ \Theta(\mathrm{T}) = -6 * \mathrm{T}$ *for* $\mathrm{T} \in \Lambda^3_2(\mathbb{R}^6)$.
(2) $\Pi_1 \circ \Theta(\mathrm{T}) = -2 * \mathrm{T}$ *for* $\mathrm{T} \in \Lambda^3_6(\mathbb{R}^6)$.
(3) $\Pi_1 \circ \Theta(\mathrm{T}) = 2 * \mathrm{T}$ *for* $\mathrm{T} \in \Lambda^3_{12}(\mathbb{R}^6)$.

Let us summarize the result of these algebraic computations.

Theorem 3.4. *Let (M^6, g, J) be an almost Hermitian manifold of type*

$$\Gamma = \Gamma_2 \oplus \Gamma_6 \oplus \Gamma_{12} \oplus \Gamma_{16}.$$

Suppose that the first three parts of the intrinsic torsion are given 3-forms in the corresponding component of $\Lambda^3(\mathbb{R}^6)$,

$$\Gamma_2 = \Theta(\mathrm{T}_2), \qquad \Gamma_6 = \Theta(X \wedge \Omega), \qquad \Gamma_{12} = \Theta(\mathrm{T}_{12}).$$

The differential and the co-differential of the Kähler form do not depend on the $\mathcal{W}_2$-component of the intrinsic torsion. Moreover, we have

$$\delta\Omega = -4X, \qquad d\Omega = -6 * \mathrm{T}_2 - 2 * (X \wedge \Omega) + 2 * \mathrm{T}_{12}.$$

3.6. The Nijenhuis tensor

The Nijenhuis tensor

$$\mathrm{N}(X, Y) := [\mathrm{J}(X), \mathrm{J}(Y)] - \mathrm{J}[X, \mathrm{J}(Y)] - \mathrm{J}[\mathrm{J}(X), Y] - [X, Y]$$

in real dimension 6 has 18 components,

$$\mathrm{N}_1 := \mathrm{N}(e_1, e_3), \qquad \mathrm{N}_2 := \mathrm{N}(e_1, e_5), \qquad \mathrm{N}_3 := \mathrm{N}(e_3, e_5),$$

and is given by

$$\begin{aligned}\mathrm{N} = {} & (e_{13} - e_{24}) \otimes \mathrm{N}_1 - (e_{23} + e_{14}) \otimes \mathrm{J}(\mathrm{N}_1) + (e_{15} - e_{26}) \otimes \mathrm{N}_2 \\ & - (e_{25} + e_{16}) \otimes \mathrm{J}(\mathrm{N}_2) + (e_{35} - e_{46}) \otimes \mathrm{N}_3 - (e_{36} + e_{45}) \otimes \mathrm{J}(\mathrm{N}_3).\end{aligned}$$

We apply the anti-symmetrization map $\Phi : \mathbb{R}^6 \otimes \mathfrak{m}^6 \to \Lambda^3(\mathbb{R}^6)$. Then $\Phi(\mathrm{N})$ is contained in $\Lambda^3_2(\mathbb{R}^6)$. Consequently, the Nijenhuis tensor is an element of the subspace $\mathcal{W}_1 \oplus \mathcal{W}_2 \subset \mathbb{R}^6 \otimes \mathfrak{m}^6$ and coincides with the $(\mathcal{W}_1 \oplus \mathcal{W}_2)$-part of the intrinsic torsion.

In particular, we obtain a characterization of Hermitian manifolds.

Theorem 3.5. *The almost complex structure* J *is integrable if and only if the* $(\mathcal{W}_1 \oplus \mathcal{W}_2)$*-part of its intrinsic torsion vanishes. The Nijenhuis tensor is totally skew-symmetric if and only if the* $\mathcal{W}_2$*-part of the intrinsic torsion vanishes.*

3.7. Differential equations characterizing the types

We identified the different parts of the intrinsic torsion with the differential and the co-differential of the Kähler form as well as with the Nijenhuis tensor. These formulas yield differential equations characterizing any type of a non-integrable Hermitian geometry. Some of these classes have special names. In general, we fix a six-dimensional almost Hermitian manifold (M^6, g, J).

Corollary 3.3. *The following conditions are equivalent*:

(1) *The structure is of type* $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$.
(2) $\delta\Omega = 0$.
(3) $\Omega \wedge d\Omega = 0$.
(4) $\mathrm{J}(d\Omega) = *d\Omega$.

Manifolds of that type are called almost semi-Kähler or co-symplectic.

Corollary 3.4. *The following conditions are equivalent*:

(1) *The structure is of type* $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_4$.
(2) $\tau^2[d\Omega - 1/2 * (\delta\Omega \wedge \Omega)] = -9[d\Omega - 1/2 * (\delta\Omega \wedge \Omega)]$.

Corollary 3.5. *The following conditions are equivalent*:

(1) *The structure is of type* $\mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$.
(2) *The Nijenhuis tensor is totally skew-symmetric*.
(3) *There exists a linear connection* ∇ *preserving the almost Hermitian structure and with totally skew-symmetric torsion*.

The equivalence of the second and third conditions has been proved in [22] (see also [19]). Almost Hermitian manifolds satisfying the latter condition are called G_1-*manifolds*.

Corollary 3.6. *The following conditions are equivalent*:

(1) *The structure is of type* $\mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$.
(2) $\tau^2(d\Omega) = -d\Omega$.

Almost Hermitian manifolds of that type are called G_2-*manifolds*.

We investigate next the almost Hermitian structures of pure type, where only one component of the intrinsic torsion does not vanish.

Corollary 3.7. *The following conditions are equivalent*:

(1) *The structure is of type* $\mathcal{W}_2$.
(2) $d\Omega = 0$.

Manifolds of that type are called almost Kähler or symplectic.

Corollary 3.8. *The following conditions are equivalent*:

(1) *The structure is of type* $\mathcal{W}_3$.
(2) J *is integrable and* $\delta\Omega = 0$.
(3) N *is totally skew-symmetric and* $\mathrm{J}(d\Omega) = *d\Omega$, $\tau^2(d\Omega) = -d\Omega$.

Almost Hermitian manifolds of that type are called semi-Kähler.

Corollary 3.9. *The following conditions are equivalent*:

(1) *The structure is of type* $\mathcal{W}_4$.
(2) N *is totally skew-symmetric and* $2d\Omega = (\delta\Omega \circ \mathrm{J}) \wedge \Omega$.

Manifolds of that type are called locally conformal Kähler.

The most interesting and rigid class of almost Hermitian manifolds in dimension 6 is the class of so-called *nearly Kähler manifolds*. In the 1060s and 1970s of the last century, they have also been called *Tachibana spaces* or K-*spaces* (see [25–27,35,36,38,40]). Nearly Kähler manifolds correspond to the pure type $\mathcal{W}_1$ and we describe this class of almost Hermitian structures in the spirit of the previous corollaries.

Corollary 3.10. *The following conditions are equivalent*:

(1) *The structure is of type* $\mathcal{W}_1$.
(2) N *is totally skew-symmetric and* $\delta\Omega = 0$, $\tau^2(d\Omega) = -9d\Omega$.

Furthermore, the differential of the Kähler form satisfies the following equations:

$$\Omega \wedge d\Omega = 0, \qquad \mathrm{J}(d\Omega) = *d\Omega.$$

There is an equivalent characterization of nearly Kähler manifolds.

Theorem 3.6. *An almost Hermitian manifold is nearly Kähler if and only if, for any vector* X,

$$(\nabla^{\mathrm{LC}}_X \Omega)(X, -) = 0.$$

Proof. Consider the U(3)-equivariant map $\mathbb{R}^6 \otimes \mathfrak{m}^6 \to \mathrm{S}^2(\mathbb{R}^6) \otimes \mathbb{R}^6$ defined by

$$\Gamma \to \hat{\Gamma}(X, Y) = (\nabla^{\mathrm{LC}}_X \Omega)(Y) + (\nabla^{\mathrm{LC}}_Y \Omega)(X).$$

It turns out that its kernel coincides with the subspace $\mathcal{W}_1 \subset \mathbb{R}^6 \otimes \mathfrak{m}^6$. □

4. The characteristic connection of a G_1-manifold

4.1. *The general formula for the characteristic connection*

An almost Hermitian manifold is of type G_1 if and only if it admits a linear connection preserving the structure with skew-symmetric torsion (see Corollary 3.5), and, in this case, the connection is unique. In this generality, this result has been proved in the paper [22]. For special types of almost Hermitian manifolds, the *characteristic connection* has been considered before. For nearly Kähler manifolds, Gray used it in 1970 (see [27, p. 304]) in order to express the Chern classes. In 1976 (see [28, p. 237]), he proved that the first Chern class of a six-dimensional nearly Kähler, non-Kähler manifold vanishes. On the other hand, the characteristic connection of a Hermitian manifold has been used by Bismut in 1989 [11] in the proof of the local index theorem. Let us compute the formula for the torsion of the characteristic connection of an almost Hermitian manifold of type G_1. Using the ansatz

$$\Gamma = \Gamma_2 \oplus \Gamma_6 \oplus \Gamma_{12}, \quad \Gamma_2 = \Theta(\mathrm{T}_2), \quad \Gamma_6 = \Theta(X \wedge \Omega), \quad \Gamma_{12} = \Theta(\mathrm{T}_{12})$$

as well as the formula $2\Gamma = -\Theta(\mathrm{T}^{\mathrm{c}})$ relating Γ and the torsion form of the characteristic connection ∇^{c} (see [19,20]), we obtain by Theorem 3.4

$$\mathrm{T}^{\mathrm{c}} = -2\mathrm{T}_2 - 2(X \wedge \Omega) - 2\mathrm{T}_{12} = -8\mathrm{T}_2 + \mathrm{J}(d\Omega).$$

The torsion form of the characteristic connection of a Hermitian manifold ($\Gamma_2 = \Gamma_{16} = 0$) is the twisted differential of the Kähler form (see [22, Theorem 10.1])

$$\mathrm{T}^{\mathrm{c}} = \mathrm{J}(d\Omega).$$

For nearly Kähler manifolds, we have $d\Omega = -6 * \mathrm{T}_2 = -6\mathrm{J}(\mathrm{T}_2)$. The torsion of the characteristic connection is again proportional to the twisted differential of the Kähler form

$$\mathrm{T}^c = -\tfrac{1}{3}\mathrm{J}(d\Omega).$$

An easy computation yields an equivalent formula for the characteristic connection and its torsion, namely

$$\mathrm{T}^c(X, Y) = -\mathrm{J}((\nabla^{LC}_X \mathrm{J})(Y)), \qquad \nabla^c_X Y = \tfrac{1}{2}(\nabla^{LC}_X Y - \mathrm{J}(\nabla^{LC}_X \mathrm{J}(Y))).$$

The latter formula is the original definition of the characteristic connection of a nearly Kähler manifold as it appears in the papers of Gray [27,28].

Combining the formula

$$\mathrm{T}^c = -2\mathrm{T}_2 - 2(X \wedge \Omega) - 2\mathrm{T}_{12}$$

with the general formula of Theorem 3.4

$$d\Omega = -6 * \mathrm{T}_2 - 2 * (X \wedge \Omega) + 2 * \mathrm{T}_{12},$$

we can express the difference $d\Omega - *\mathrm{T}^c$,

$$d\Omega - *\mathrm{T}^c = 4 \cdot (*\mathrm{T}_{12} - *\mathrm{T}_2).$$

Consequently, we obtain the following proposition.

Proposition 4.1. *The characteristic torsion form* T^c *of a* G_1*-manifold is co-closed,* $\delta(\mathrm{T}^c) = 0$, *if and only if*

$$d * \mathrm{T}_2 = d * \mathrm{T}_{12}.$$

In particular, any almost Hermitian manifold of pure type $\mathcal{W}_1$, *of pure type* $\mathcal{W}_3$ *or of pure type* $\mathcal{W}_4$ *has a co-closed characteristic torsion form.*

4.2. The characteristic connection of a nearly Kähler manifold

Nearly Kähler manifolds in dimension 6 have certain special properties. They are Einstein spaces of positive scalar curvature, the almost complex structure is never integrable, the first Chern class vanishes and they admit a spin structure (see [28]). Moreover, nearly Kähler manifolds in dimension 6 are exactly those Riemannian spaces admitting real Riemannian Killing spinors (see [17,18,21,30]). From our point of view, one of the interesting properties of nearly Kähler 6-manifolds is the ∇^c-parallelism of their torsion form T^c. This is a consequence of certain curvature identities already proved by Takamatsu [38], Matsumoto [35] and Gray [28]. In Kirichenko's paper [33], the ∇^c-parallelism of T^c appeared probably for the first time explicitly. We will outline a simple proof of this theorem. A nearly Kähler structure is characterized by the conditions

$$Z = Z^* \oplus \Gamma_2, \quad \Gamma_2 = \Theta(\mathrm{T}_2), \;\; \mathrm{T}_2 \in \Lambda^3_2.$$

The derivative of the Kähler form and the characteristic torsion are given by the formulas

$$d * \Omega = 0, \qquad d\Omega = -6 * \mathrm{T}_2, \qquad \mathrm{T}^{\mathrm{c}} = -2\mathrm{T}_2, \qquad \mathrm{T}^{\mathrm{c}}(X, Y) = -\mathrm{J}(\nabla^{\mathrm{LC}}_X \mathrm{J})(Y).$$

A nearly Kähler 6-manifold is of constant type in the sense of Gray [27], i.e.,

$$\|\mathrm{J}(\nabla^{\mathrm{LC}}_X \mathrm{J})(Y)\|^2 = \frac{\mathrm{Scal}}{30}\{\|X\|^2\|Y\|^2 - g^2(X, Y) - g^2(X, \mathrm{J}(Y))\}.$$

In particular, the length of the characteristic torsion coincides with the scalar curvature

$$\|\mathrm{T}^{\mathrm{c}}\|^2 = \frac{2}{15}\mathrm{Scal}.$$

Since a nearly Kähler 6-manifold is Einstein, the length of the characteristic torsion is hence constant. It is a remarkable fact that this property of the characteristic connection implies alone that it is parallel.

Theorem 4.1. *The torsion of the characteristic connection of a nearly Kähler* 6*-manifold is parallel*

$$\nabla^{\mathrm{c}}\mathrm{T}^{\mathrm{c}} = 0.$$

The characteristic connection of a six-dimensional nearly Kähler non-Kähler manifold is a $\mathrm{SU}(3)$*-connection.*

Proof. First of all, we remark that, for a 3-form $\mathrm{T}_2 \in \Lambda^3_2$, we have $\Gamma_2(X) = X \lrcorner \mathrm{T}_2$. This implies that the characteristic connection coincides with the connection Z^* in the decomposition $Z = Z^* \oplus \Gamma_2$ of the Levi–Civita connection. The characteristic connection induces a metric covariant derivative in the two-dimensional bundle Λ^3_2. Since T_2 has constant length, there exists a 1-form A such that

$$\nabla^{\mathrm{c}}_X \mathrm{T}_2 = A(X) \cdot (*\mathrm{T}_2).$$

The co-differentials $\delta(\mathrm{T}) = \delta^{\nabla}(\mathrm{T})$ of the torsion form of a metric connection coincide (see [4]). Therefore we obtain

$$0 = \delta(*d\Omega) = \delta(6\mathrm{T}_2) = -3\delta(\mathrm{T}^{\mathrm{c}}) = -3\delta^{\nabla^{\mathrm{c}}}(\mathrm{T}^{\mathrm{c}}) = 6\delta^{\nabla^{\mathrm{c}}}(\mathrm{T}_2) = 6A \lrcorner (*\mathrm{T}_2).$$

The algebraic type $\mathrm{T}_2 \in \Lambda^3_2$ implies that $A = 0$ vanishes. □

We remark that the Kähler form of a nearly Kähler 6-manifold is an eigenform of the Hodge–Laplace operator. Indeed, we can write the equation $\nabla^{\mathrm{c}}\mathrm{T}^{\mathrm{c}} = 0$ equivalently as

$$\nabla^{\mathrm{LC}}_X(*d\Omega) + \frac{\mathrm{Scal}}{10}\mathrm{J}(X) \wedge \Omega = 0.$$

The latter formula immediately implies that $5 \cdot \Delta\Omega = 2 \cdot \mathrm{Scal} \cdot \Omega$.

Remark 4.1. The equation (see [40, pp. 146–149] or [28, Theorem 5.2])

$$\sum_{i=1}^{6} g(\mathrm{R}(X, Y)e_i, \mathrm{J}(e_i)) = -\frac{\mathrm{Scal}}{15}\Omega(X, Y)$$

is equivalent to the fact that the characteristic connection is an $\mathrm{SU}(3)$-connection. Indeed, the structure equation reads as

$$\Omega^{\mathrm{LC}} = \Omega^{Z^*} + d\Gamma_2 + [Z^*, \Gamma_2] + \tfrac{1}{2}[\Gamma_2, \Gamma_2].$$

We project onto the central element of the Lie algebra $\mathfrak{u}(3)$. Since $d\Gamma_2$ and $[Z^*, \Gamma_2]$ have values in the subspace $\mathfrak{m}^6$, we obtain

$$\mathrm{pr}(\Omega^{\mathrm{LC}}) = \mathrm{pr}(\Omega^{Z^*}) + \mathrm{pr}(\tfrac{1}{2}[\Gamma_2, \Gamma_2]).$$

The curvature identity of a nearly Kähler manifold mentioned above as well as $15\|\mathrm{T}^{\mathrm{c}}\|^2 = 2\mathrm{Scal}$ yield that

$$\mathrm{pr}(\Omega^{\mathrm{LC}}) = \mathrm{pr}(\tfrac{1}{2}[\Gamma_2, \Gamma_2]),$$

i.e., the characteristic connection is an $\mathrm{SU}(3)$-connection.

Remark 4.2. The complete nearly Kähler manifolds with characteristic holonomy group contained in $\mathrm{U}(2) \times \mathrm{U}(1) \subset \mathrm{U}(3)$ have been classified in [10]. There are only two spaces of that type, namely the projective space $\mathbb{CP}^3$ and the flag manifold $\mathrm{F}(1, 2)$ equipped with their homogeneous (non-Kähler) nearly Kähler structure. However, there is another interesting case. The three-dimensional complex irreducible representation of the group $\mathrm{SU}(2)/\{\pm 1\}$ is *reducible* as a real representation (see the discussion after Theorem 4.4). It is realized as the characteristic holonomy by a left-invariant nearly Kähler structure on the Lie group $\mathrm{S}^3 \times \mathrm{S}^3$.

Remark 4.3. Homogeneous nearly Kähler manifolds have been classified in [12]. The geometry of these examples has been described in detail in [8].

4.3. G_1*-manifolds with parallel torsion and non-vanishing divergence*

The aim of the next two sections is to study the structure of almost Hermitian manifolds with a ∇^{c}-parallel characteristic torsion T^{c}. We already know that any nearly Kähler manifold has this property. Moreover, naturally reductive, almost Hermitian manifolds are automatically of type G_1 and their torsion form is ∇^{c}-parallel, too. Indeed, the canonical connection ∇^{can} of a naturally reductive space has totally skew-symmetric torsion and preserves the almost Hermitian structure. Since these two properties single out the characteristic connection of the almost Hermitian structure, we conclude that ∇^{c} and ∇^{can} coincide. But the canonical connection of any naturally reductive space has parallel torsion. This series of examples includes compact Lie groups equipped with a left-invariant almost Hermitian structure. On the other side, left-invariant almost complex structures on nilmanifolds in dimension 6 have been discussed in detail in the papers [1,16]. Here the torsion form is, in general, not parallel.

In this section we study G_1-manifolds with a ∇^{c}-parallel torsion form and non-vanishing divergence of the Kähler form. The intrinsic torsion of a G_1-manifold is given by two 3-forms T_2, T_{12} and a 1-form X. The equations are

$$d\Omega = -6 * \mathrm{T}_2 - 2 * (X \wedge \Omega) + 2 * \mathrm{T}_{12}, \qquad \mathrm{T}^{\mathrm{c}} = -2\mathrm{T}_2 - 2(X \wedge \Omega) - 2\mathrm{T}_{12}.$$

Since $\nabla^{\mathrm{c}}\mathrm{T}^{\mathrm{c}} = 0$ implies $\delta\mathrm{T}^{\mathrm{c}} = \delta^{\nabla^{\mathrm{c}}}\mathrm{T}^{\mathrm{c}} = 0$, we obtain the necessary conditions

$$d(*\mathrm{T}_{12} - *\mathrm{T}_2) = 0, \qquad d(*(X \wedge \Omega) + 2 * \mathrm{T}_2) = 0.$$

The characteristic connection preserves the splitting $\Lambda^3 = \Lambda^3_2 \oplus \Lambda^3_6 \oplus \Lambda^3_{12}$. Therefore, the condition $\nabla^{\mathrm{c}}\,\mathrm{T}^{\mathrm{c}} = 0$ is equivalent to

$$\nabla^{\mathrm{c}} X = 0, \qquad \nabla^{\mathrm{c}}\mathrm{T}_2 = 0, \qquad \nabla^{\mathrm{c}}\mathrm{T}_{12} = 0.$$

The forms $\mathrm{S}_1 := \mathrm{T}_{12} - \mathrm{T}_2$ and $\mathrm{S}_2 := X \wedge \Omega + 2\mathrm{T}_{12}$ are ∇^{c}-parallel and divergence-free

$$\nabla^{\mathrm{c}}\mathrm{S}_1 = 0 = \nabla^{\mathrm{c}}\mathrm{S}_2, \qquad \delta\mathrm{S}_1 = 0 = \delta\mathrm{S}_2.$$

Using the formula in [4, Proposition 5.1] we conclude that ($\alpha = 1, 2$)

$$\sum_{i,j=1}^{6} (e_i \lrcorner\, e_j \lrcorner\, \mathrm{T}^{\mathrm{c}}) \wedge (e_i \lrcorner\, e_j \lrcorner\, \mathrm{S}_\alpha) = 0.$$

The latter equation couples the 3-forms T_2 and T_{12} via the form X. Lengthily, but elementary computations allow us to express this link directly.

Proposition 4.2. *Let M^6 be a G_1-manifold with parallel characteristic torsion, $\nabla^{\mathrm{c}}\mathrm{T}^{\mathrm{c}} = 0$. Then, for any vectors Y, Z, the following equations are satisfied*:

$$\mathrm{T}_{12}(X, \mathrm{J}X, Y) = 0, \qquad \mathrm{T}_{12}(X, Y, Z) = \mathrm{T}_{12}(X, \mathrm{J}Y, \mathrm{J}Z) + 2\mathrm{T}_2(X, Y, Z),$$
$$\mathrm{T}_{12}(\mathrm{J}X, Y, Z) = \mathrm{T}_{12}(\mathrm{J}X, \mathrm{J}Y, \mathrm{J}Z) - 2\mathrm{T}_2(\mathrm{J}X, Y, Z).$$

The vector fields X and $\mathrm{J}X$ are ∇^{c}-parallel and we compute their commutator

$$[X, \mathrm{J}X] = -\mathrm{T}^{\mathrm{c}}(X, \mathrm{J}X, -).$$

For algebraic reasons, we have $\mathrm{T}_2(X, \mathrm{J}X-) = 0$ and the formula simplifies

$$[X, \mathrm{J}X] = 2\mathrm{T}_{12}(X, \mathrm{J}X, -).$$

The first equation of this proposition yields now the proof of the following corollary.

Corollary 4.1. *Let M^6 be a G_1-manifold with ∇^{c}-parallel characteristic torsion. Then X and $\mathrm{J}X$ are commuting Killing vector fields of constant length*

$$[X, \mathrm{J}X] = 0.$$

In case that the vector field $X \neq 0$ is non-trivial, the leaves of the integrable distribution $\{X, \mathrm{J}X\}$ are two-dimensional flat and totally geodesic submanifolds. They are the orbits of an isometric $\mathbb{R}^2$-action.

From now on we assume that the vector field $X \neq 0$ is non-trivial. Then the tangent bundle splits into the integrable distribution $T^{\mathrm{v}} = \mathrm{Lin}(X, \mathrm{J}X)$ and its orthogonal complement T^{h}. We decompose the 3-forms T_2 and T_{12} into

$$\mathrm{T}_2 = X \wedge \Omega_1 + \mathrm{J}X \wedge \Omega_2, \qquad \mathrm{T}_{12} = X \wedge \Omega_3 + \mathrm{J}X \wedge \Omega_4,$$

$\Omega_1, \ldots, \Omega_4 \in \Lambda^2(T^{\mathrm{h}})$ are horizontal 2-forms. Remark that, for purely algebraic reasons, these forms are orthogonal to the horizontal Kähler form $e_3 \wedge e_4 + e_5 \wedge e_6$. Proposition 4.2 can be reformulated as

$$\Omega_3 = \mathrm{J}(\Omega_3) + 2\Omega_1, \qquad \Omega_4 = \mathrm{J}(\Omega_4) + 2\Omega_2,$$

and all these forms are ∇^{c}-parallel. The next proposition summarizes the result of a straightforward calculation.

Proposition 4.3. *The Lie derivative of the Kähler form and the differentials of the forms X and $\mathrm{J}X$ are given by*

$$dX = \|X\|^2(\mathrm{J}(\Omega_3) - 3\Omega_3 - 2\Omega) + 2X \wedge \mathrm{J}X,$$
$$d\mathrm{J}X = \|X\|^2(\mathrm{J}(\Omega_4) - 3\Omega_4), \qquad \mathcal{L}_X\Omega = 8\|X\|^2\Omega_2.$$

A direct consequence of these formulas is as follows.

Theorem 4.2. *Let (M^6, g, J) be an almost Hermitian 6-manifold of type $\mathcal{W}_1 \oplus \mathcal{W}_4$. If the torsion of its characteristic connection is parallel, $\nabla^{\mathrm{c}}\mathrm{T}^{\mathrm{c}} = 0$, then M^6 is either of pure type $\mathcal{W}_1$ or of pure type $\mathcal{W}_4$.*

Remark 4.4. The characteristic torsion of a nearly Kähler manifold is ∇^{c}-parallel. On the other hand, suppose that M^6 is of pure type $\mathcal{W}_4$ and X is ∇^{c}-parallel. Then we obtain

$$0 = \nabla^{\mathrm{c}}_Z \mathrm{J}X = \nabla^{\mathrm{LC}}_Z \mathrm{J}X, \qquad dX = 2(-\|X\|^2\,\Omega + X \wedge \mathrm{J}X).$$

The vector field $\mathrm{J}X$ is ∇^{LC}-parallel, i.e., the manifold is a generalized Hopf manifold. Up to a scaling of the length of X, the manifold is locally isometric to a product $M^6 = N^5 \times \mathbb{R}^1$ of the line $\mathbb{R}^1$ by a five-dimensional Sasakian manifold N^5. Conversely, any product of a Sasakian manifold by $\mathbb{R}^1$ is an almost Hermitian manifold of type $\mathcal{W}_4$ with parallel torsion. These $\mathcal{W}_4$-manifolds have been studied by Vaisman [39].

We consider now Hermitian manifolds. The complex structure is integrable and the forms Ω_1 and Ω_2 vanish. The forms $\Omega_3, \Omega_4 \in \Lambda^2_-(T^{\mathrm{v}})$ are anti-self-dual with respect to the four-dimensional horizontal Hodge operator and the formulas of Proposition 4.3 simplify,

$$dX = -2\|X\|^2\Omega_3 - 2\|X\|^2\Omega + 2X \wedge \mathrm{J}X, \qquad d\mathrm{J}X = -2\|X\|^2\Omega_4.$$

Furthermore, we obtain immediately that

$$\mathcal{L}_X \Omega = 0, \qquad \mathcal{L}_{\mathrm{J}X} \Omega = 0, \qquad d(\|X\|^2 \Omega - X \wedge \mathrm{J}X) = 0.$$

The forms Ω_3 and Ω_4 are closed. Suppose that the Killing vector fields X and $\mathrm{J}X$ induce a regular group action, i.e., the orbit space $\hat{X}^4$ is smooth. Then $\hat{X}^4$ admits a Riemannian metric $\hat{g}$ and a complex structure $\hat{\mathrm{J}}$ with Kähler form

$$\hat{\Omega} = \Omega - \frac{1}{\|X\|^2} X \wedge \mathrm{J}X.$$

In particular, $\hat{X}^4$ is a four-dimensional Kähler manifold. The forms Ω_3 and Ω_4 project onto $\hat{X}^4$. A direct computation yields that the ∇^{c}-parallelism of these forms on M^6 can be reformulated as the condition that their projections $\hat{\Omega}_3$ and $\hat{\Omega}_4$ are anti-self-dual and parallel forms on the Kähler manifold $\hat{X}^4$,

$$\hat{\nabla}\hat{\Omega}_3 = 0, \qquad \hat{\nabla}\hat{\Omega}_4 = 0, \qquad *\hat{\Omega}_3 = -\hat{\Omega}_3, \qquad *\hat{\Omega}_4 = -\hat{\Omega}_4.$$

The structure group of the principal fiber bundle $M^6 \to \hat{X}^4$ is two-dimensional and Abelian. Up to a scaling of the length, the pair $\{X, \mathrm{J}X\}$ is a connection. Its curvature is the pair of 2-forms $(\mathrm{Curl}_1, \mathrm{Curl}_2)$ on $\hat{X}^4$ given by the differentials of X and $\mathrm{J}X$, i.e.,

$$\mathrm{Curl}_1 = -2\hat{\Omega}_3 - 2\hat{\Omega}, \qquad \mathrm{Curl}_2 = -2\hat{\Omega}_4.$$

Vice versa, we can reconstruct the whole six-dimensional structure out of the four-dimensional Kähler manifold $(\hat{X}^4, \hat{g}, \hat{\mathrm{J}})$ and the two parallel forms $\hat{\Omega}_3, \hat{\Omega}_4 \in \Lambda^2_-(\hat{X}^4)$. In the compact case we need that $2\hat{\Omega}_4$ and $2\hat{\Omega} + 2\hat{\Omega}_3$ are curvature forms of some $\mathrm{U}(1)$-bundle, i.e.,

$$2\hat{\Omega}_4, 2\hat{\Omega} + 2\hat{\Omega}_3 \in \mathrm{H}^2(\hat{X}^4; \mathbb{Z}).$$

We summarize the result for compact Hermitian spaces M^6.

Theorem 4.3. *The compact regular Hermitian manifolds* (M^6, g, J) *with non-vanishing divergence* $-4X = \delta\Omega \neq 0$ *of the Kähler form and* ∇^{c}*-parallel characteristic torsion* T^{c} *correspond to triples* $(\hat{X}^4, \hat{\Omega}_3, \hat{\Omega}_4)$ *consisting of a compact four-dimensional Kähler manifold* $\hat{X}^4$ *and two parallel anti-self-dual forms* $\hat{\Omega}_3, \hat{\Omega}_4$ *such that*

$$2\hat{\Omega}_4, 2\hat{\Omega} + 2\hat{\Omega}_3 \in \mathrm{H}^2(\hat{X}^4; \mathbb{Z}).$$

It is easy to describe the possible Kähler manifolds $\hat{X}^4$. First of all, a parallel anti-self-dual 2-form gives rise to a parallel complex structure of opposite orientation. Then a compact four-dimensional space with two independent parallel complex structures with equal orientation is hyperKähler. The existence of opposite parallel complex structures restricts it to be a torus (see [31]). Since toric bundles over tori are always two-step nilmanifolds, the six-dimensional manifold is at least diffeomorphic to a locally homogeneous space. When the forms $\hat{\Omega}_3$ and $\hat{\Omega}_4$ are linearly depent, M^6 is actually a product $\mathrm{S}^1 \times N^5$ and N^5 is a S^1-bundle over a 4-manifold which is covered by a product of two surfaces.[1]

[1] The authors thank the referee for a hint completing this classification.

Remark 4.5. Hermitian structures with an SU(3)-holonomy of the characteristic connection have been constructed recently on certain toric bundles (see [24]). The condition $\nabla^c T^c = 0$ is a stronger condition and, consequently, our family is much smaller.

4.4. $\mathcal{W}_3$-manifolds with parallel torsion

An interesting problem is the structure of $\mathcal{W}_3$-manifolds with ∇^c-parallel torsion. The equations characterizing these Hermitian manifolds are (see [22])

$$d\Omega = 2 * T_{12} = - * T^c, \qquad \delta\Omega = 0, \qquad \nabla^c T^c = 0, \qquad dT^c = 2\sigma_{T^c},$$

where the 4-form σ_{T^c} is defined by the formula

$$\sigma_{T^c} := \frac{1}{2} \sum_{i=1}^{6} (e_i \lrcorner T^c) \wedge (e_i \lrcorner T^c).$$

We remark that in the class of Hermitian $\mathcal{W}_3$-manifolds an analogue of Theorem 4.1 does not hold.

Example 4.1. Consider the three-dimensional complex Heisenberg group. There exists a left-invariant metric with the following structure equations:

$$de_1 = de_2 = de_3 = de_4 = 0, \qquad de_5 = e_{13} - e_{24}, \qquad de_6 = e_{14} + e_{23}.$$

The differential of the Kähler form is given by

$$d\Omega = e_{136} - e_{246} - e_{145} - e_{235}.$$

Consequently, the Hermitian structure is of pure type $\mathcal{W}_3$ and its torsion is given by $T^c = e_{245} - e_{135} - e_{236} - e_{146}$. We compute the derivative dT^c and the 4-form σ_{T^c},

$$dT^c = -4e_{1234}, \qquad \sigma_{T^c} = 2e_{1234} - (e_{12} + e_{34}) \wedge e_{56}.$$

Since $dT^c \neq 2\sigma_{T^c}$, the torsion form of the Heisenberg group is not parallel.

The U(3)-orbit type of the parallel torsion form $T^c \in \Lambda^3_{12}$ is constant. There are only two types of 3-forms in Λ^3_{12} with a non-Abelian isotropy group.

Theorem 4.4. *Let* $T \in \Lambda^3_{12}$ *be a* 3*-form and denote by* $G_T \subset U(3)$ *the connected component of its isotropy group. If the dimension of* G_T *is at least* 3, *then one of the following two cases occurs*:

(1) *The group* G_T *is isomorphic to* U(2) *and the embedding into* U(3) *is given by the homomorphism*

$$G_T = \left\{ \begin{bmatrix} g & 0 \\ 0 & \det(g) \end{bmatrix}, g \in U(2) \right\}.$$

Up to a complex factor, there exists one orbit of that type represented by the 3-form

$$\mathrm{T} = (e_{135} - e_{245} + e_{236} + e_{146}).$$

(2) *The group* $\mathrm{G_T}$ *is isomorphic to* $\mathrm{SU}(2)/\{\pm 1\} = \mathrm{SO}(3)$ *and the embedding into* $\mathrm{U}(3)$ *is the unique three-dimensional irreducible complex representation of* $\mathrm{SU}(2)$. *Up to a complex factor, there exists one orbit of that type represented by the* 3*-form*

$$\mathrm{T} = 2(e_{123} - e_{356}) - (e_{246} + e_{136}) + (e_{145} - e_{235}).$$

Proof. We use the explicit equations defining the Lie algebra $\mathfrak{g}_\mathrm{T} \subset \mathfrak{u}(3)$ of the isotropy group $\mathrm{G_T}$. The 3-form T depends on 12 real parameters,

$$\begin{aligned}\mathrm{T} = {} & A_1(e_{123} - e_{356}) + A_2(e_{124} - e_{456}) + A_3(e_{125} - e_{345}) + A_4(e_{126} - e_{346}) \\ & + A_5(e_{134} - e_{156}) + A_6(e_{234} - e_{256}) + A_7(e_{135} + e_{245}) \\ & + A_8(e_{246} + e_{136}) + A_9(e_{145} - e_{235}) + A_{10}(e_{236} - e_{146}) \\ & + A_{11}(e_{135} - e_{245} + e_{236} + e_{146}) + A_{12}(e_{246} - e_{136} + e_{145} + e_{235}).\end{aligned}$$

An arbitrary 2-form in $\mathfrak{u}(3)$ depends on nine real parameters

$$\begin{aligned}\omega = {} & \omega_{12}e_{12} + \omega_{13}(e_{13} + e_{24}) + \omega_{14}(e_{14} - e_{23}) + \omega_{15}(e_{15} + e_{26})\omega_{16}(e_{16} - e_{25}) \\ & + \omega_{34}e_{34} + \omega_{35}(e_{35} + e_{46}) + \omega_{36}(e_{36} - e_{45}) + \omega_{56}e_{56}.\end{aligned}$$

The condition $\varrho_*(\omega)\mathrm{T} = 0$ is a linear system of 12 equations with respect to nine variables ω_{ij} given by the following (12×9)–matrix $\mathcal{A}_\mathrm{T}$:

$$\begin{bmatrix}
2A_{12} & 0 & 0 & 2A_2 & -2A_1 & 2A_{12} & -2A_6 & 2A_5 & -2A_{12} \\
-2A_{11} & 0 & 0 & -2A_1 & -2A_2 & -2A_{11} & 2A_5 & 2A_6 & 2A_{11} \\
A_6 & A_1 & A_2 & -A_3 & -A_4 & 0 & D - 2A_{12} & -B - 2A_{11} & 0 \\
-A_5 & A_2 & -A_1 & -A_4 & A_3 & 0 & -B + 2A_{11} & -D - 2A_{12} & 0 \\
0 & A_5 & -A_6 & C - 2A_{12} & -A - 2A_{11} & -A_2 & -A_3 & -A_4 & 0 \\
0 & -A_6 & -A_5 & A - 2A_{11} & C + 2A_{12} & -A_1 & A_4 & -A_3 & 0 \\
-D & 2A_4 & -2A_3 & 0 & 0 & D & -2A_6 & -2A_5 & -D \\
B & -2A_3 & -2A_4 & 0 & 0 & -B & 2A_5 & -2A_6 & B \\
0 & 2A_{10} & 2A_8 & -A_6 & -A_5 & 0 & A_2 & A_1 & A_3 \\
0 & -2A_9 & 2A_7 & -A_5 & A_6 & 0 & A_1 & -A_2 & -A_4 \\
C & -2A_4 & -2A_3 & 2A_2 & 2A_1 & -C & 0 & 0 & -C \\
A & -2A_3 & 2A_4 & 2A_1 & -2A_2 & -A & 0 & 0 & -A
\end{bmatrix}.$$

We introduced the notion $A := A_7 + A_{10}$, $B := A_7 - A_{10}$, $C := A_8 + A_9$, $D := A_8 - A_9$. If $\mathrm{T} \neq 0$, the rank of this matrix is at least 3. Therefore, the dimension of the Lie algebra is bounded by $\dim(\mathfrak{g}_\mathrm{T}) \leq 6$. Since $\mathrm{T} \in \Lambda^3_{12}$, the central element $\Omega \in \mathfrak{u}(3)$ does not belong to $\mathfrak{g}_\mathrm{T}$. An elementary discussion concerning subgroups of $\mathrm{U}(3)$ yields the result that the group $\mathrm{G_T}$ is conjugated to $\mathrm{SO}(3)$ or $\mathrm{U}(2)$ and realized in the way as the theorem states. On the other side, given one of these two Lie algebras $\mathfrak{g}_\mathrm{T}$, the matrix $\mathcal{A}_\mathrm{T}$ computes the corresponding 3-form T up to a complex factor. □

First, we study the case of $G_{T^c} = U(2)$. Then the 2-forms $e_{12} + e_{34}$ and e_{56} are globally defined and ∇^c-parallel,

$$\nabla^c(e_{12} + e_{34}) = 0 = \nabla^c(e_{56}).$$

Using [4, Proposition 5.2], we compute the exterior derivative $d(e_{56})$,

$$d(e_{56}) = \sum_{i=1}^{6}(e_i \lrcorner e_{56}) \wedge (e_i \lrcorner T^c) = T^c.$$

Moreover, $d\Omega = - * T^c$ implies a formula for the derivative of the second invariant 2-form, $d(e_{12}+e_{34}) = -2 * T^c$. Let us introduce a new almost complex structure $\hat{J}$ by the condition

$$\hat{\Omega} = -(e_{12} + e_{34}) + e_{56}.$$

Then we have

$$\nabla^c\hat{\Omega} = 0, \quad d\hat{\Omega} = 3 * T^c,$$

i.e., the manifold $(M^6, g, \hat{J})$ is nearly Kähler, ∇^c is its characteristic connection, and the holonomy $\mathrm{Hol}(\nabla^c) = U(2) = G_T$ is not the whole group $SU(3)$. In the compact case these nearly Kähler manifolds have been classified in [10]. There are only two of them, namely the twistor spaces of the four-dimensional sphere or of the complex projective plain equipped with their canonical non-integrable almost complex structure and their canonical non-Kähler Einstein metric. Replacing again the almost complex structure $\hat{J}$ by J, we obtain a complete classification of all $\mathcal{W}_3$-manifolds with parallel characteristic torsion of type $G_{T^c} = U(2)$.

Theorem 4.5. *Let (M^6, g, J) be a complete Hermitian manifold of type $\mathcal{W}_3$ such that*

$$\nabla^c T^c = 0, \qquad G_{T^c} = U(2).$$

Then M^6 is the twistor space of a four-dimensional, compact self-dual Einstein manifold with positive scalar curvature. The complex structure J is the standard one of the twistor space and the metric g is the unique non-Kähler Einstein metric in the canonical 1-parameter family of metrics of the twistor space.

Remark 4.6. The latter theorem holds locally and in higher dimensions too (see [6]). In dimension 6, there are only two compact Kählerian twistor spaces, namely the projective space $\mathbb{C}P^3$ and the flag manifold $F(1, 2)$ (see [23,32]).

The second case $G_{T^c} = SU(2)/\{\pm 1\} \subset U(3)$ corresponds to the three-dimensional *complex irreducible* representation. The underlying real representation in $\mathbb{C}^3 = \mathbb{R}^6$ is *reducible*, i.e., under the action of the group G_{T^c} the Euclidean space $\mathbb{R}^6$ splits into two real and three-dimensional Lagrangian subspaces. The holonomy representation is the sum of two faithful representations. The results of [14, Lemmas 4.4 and 5.6] yield that M^6 is a so-called Ambrose–Singer manifold, i.e., the curvature R^{c} of the characteristic connection is ∇^{c}-parallel,

$$\nabla^{\mathrm{c}}\mathrm{T}^{\mathrm{c}} = 0, \qquad \nabla^{\mathrm{c}}\mathrm{R}^{\mathrm{c}} = 0.$$

Since the universal covering of $\mathrm{G}_{\mathrm{T}^{\mathrm{c}}}$ is compact, the Ambrose–Singer manifold is regular and locally isometric to a homogeneous space $\mathrm{G}/\mathrm{G}_{\mathrm{T}^{\mathrm{c}}}$. The Lie algebra of the automorphism group G is the vector space $\mathfrak{g} := \mathfrak{g}_{\mathrm{T}^{\mathrm{c}}} \oplus \mathbb{R}^6$ equipped with the bracket (see [14, Theorem 5.10])

$$[A + X, B + Y] = ([A, B] - \mathrm{R}^{\mathrm{c}}(X, Y)) + (A \cdot Y - B \cdot X - \mathrm{T}^{\mathrm{c}}(X, Y)).$$

In order to find the automorphism group as well as the Hermitian manifold, we consider the Lie subalgebra $\mathfrak{so}(3) \subset \mathfrak{so}(6)$. It is generated by the following 2-forms:

$$\omega_1 := \frac{1}{\sqrt{2}}(e_{12} - e_{56}), \qquad \omega_2 := \tfrac{1}{2}(e_{13} + e_{24} + e_{35} + e_{46}),$$
$$\omega_3 := \tfrac{1}{2}(e_{14} - e_{23} + e_{36} - e_{45}),$$

and the SO(3)-invariant form $\mathrm{T}^{\mathrm{c}} \in \Lambda^3(\mathbb{R}^6)$ is given by the formula

$$\mathrm{T}^{\mathrm{c}} := 2(e_{123} - e_{356}) - (e_{246} + e_{136} - e_{145} + e_{235}).$$

The curvature tensor of the characteristic connection is an SO(3)-invariant 2-form with values in the Lie algebra $\mathfrak{so}(3)$. Since the SO(3)-representation $\Lambda^2(\mathbb{R}^6)$ splits into $3 \cdot \mathbb{R}^3 \oplus \mathbb{R}^1 \oplus \mathrm{S}^2_0(\mathbb{R}^3)$, the curvature tensor depends a priori on three parameters. However, the first Bianchi identity yields that R^{c} is *unique*.

Lemma 4.1. *The curvature of the characteristic connection is proportional to the orthogonal projection onto* $\mathfrak{so}(3)$,

$$\mathrm{R}^{\mathrm{c}} : \Lambda^2(\mathbb{R}^6) = \mathfrak{so}(6) \to \mathfrak{so}(3), \qquad \mathrm{R}^{\mathrm{c}}(X, Y) = 4 \cdot \mathrm{pr}_{\mathfrak{so}(3)}(X \wedge Y).$$

We remark that the 3-form T^{c} satisfies the necessary condition in order to define an extension of the Lie algebra $\mathfrak{so}(3)$, namely the element of the Clifford algebra $\mathfrak{Cliff}(\mathbb{R}^6)$,

$$(\mathrm{T}^{\mathrm{c}})^2 + 4 \cdot (\omega_1^2 + \omega_2^2 + \omega_3^2)$$

is a scalar (see [37, Chapter 10.4]). It turns out that the automorphism group is isomorphic to the semi-simple Lie group $\mathrm{G} = \mathrm{SL}(2, \mathbb{C}) \times \mathrm{SU}(2)$. The Hermitian manifold $M^6 = \mathrm{G}/\mathrm{G}_{\mathrm{T}^{\mathrm{c}}}$ is a left-invariant Hermitian structure on $\mathrm{SL}(2, \mathbb{C})$ represented as a naturally reductive space by the help of the subgroup $\mathrm{SU}(2) \subset \mathrm{SL}(2, \mathbb{C})$ (see [3,7]). Since the characteristic connection of the Hermitian manifold is unique, it coincides with the canonical connection of the naturally reductive space.

Theorem 4.6. *Any Hermitian* 6*-manifold of type* $\mathcal{W}_3$ *and*

$$\nabla^{\mathrm{c}}\mathrm{T}^{\mathrm{c}} = 0, \qquad \mathrm{G}_{\mathrm{T}^{\mathrm{c}}} = \mathrm{SU}(2)/\{\pm 1\}$$

is locally isomorphic to the left-invariant Hermitian structure on the Lie group $\mathrm{SL}(2, \mathbb{C})$.

We briefly describe the Hermitian structure under consideration. Let us decompose the Lie algebra $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{su}(2)$,

$$\mathfrak{g} = \{(A, B) \in \mathcal{M}(2, \mathbb{C}) \oplus \mathcal{M}(2, \mathbb{C}) : \mathrm{tr}(A) = 0,\ B + \bar{B}^{\mathrm{t}} = 0, \mathrm{tr}(B) = 0\},$$

into the subalgebra $\mathfrak{h} := \{(B, B) \in \mathfrak{g} : B + \bar{B}^{\mathrm{t}} = 0, \mathrm{tr}(B) = 0\}$ and its complement,

$$\mathfrak{m} := \{(A, B) \in \mathfrak{g} : A - \bar{A}^{\mathrm{t}} = 0, \mathrm{tr}(A) = 0,\ B + \bar{B}^{\mathrm{t}} = 0, \mathrm{tr}(B) = 0\}.$$

The decomposition is reductive, $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$. Moreover, we introduce a complex structure $\mathrm{J} : \mathfrak{m} \to \mathfrak{m}$ as well as a scalar product $\langle\ ,\ \rangle_{\mathfrak{m}}$ by the formulas

$$\mathrm{J}(A, B) := (i \cdot B, i \cdot A), \qquad \langle (A, B), (A_1, B_1)\rangle_{\mathfrak{m}} := \mathrm{tr}(A \cdot \bar{A}_1^{\mathrm{t}}) + \mathrm{tr}(B \cdot \bar{B}_1^{\mathrm{t}}).$$

Both are $\mathfrak{h} = \mathfrak{su}(2)$-invariant. They define an almost Hermitian structure on $M^6 = (\mathrm{SL}(2, \mathbb{C}) \times \mathrm{SU}(2))/\mathrm{SU}(2) = \mathrm{SL}(2, \mathbb{C})$. It turns out that the almost complex structure is integrable and the Hermitian structure is of type $\mathcal{W}_3$ ($\delta\Omega = 0$). Its characteristic torsion form coincides with the canonical torsion of the naturally reductive space. The manifold realizes the orbit type $\mathrm{G}_{\mathrm{T^c}} = \mathrm{SU}(2)/\{\pm 1\}$. Finally, let us describe some geometric data. The Ricci tensor of the characteristic connection is proportional to the metric,

$$\mathrm{Ric}^{\nabla^{\mathrm{c}}} = -\tfrac{1}{3} \cdot \|\mathrm{T^c}\|^2 \cdot \mathrm{Id}.$$

The 3-form $\mathrm{T^c}$ acts on the spinor bundles $S^{\pm}$ with a one-dimensional kernel and there exist two ∇^{c}-parallel spinor fields $\Psi^{\pm}$,

$$\nabla^{\mathrm{c}}\Psi^{\pm} = 0, \qquad \mathrm{T^c} \cdot \Psi^{\pm} = 0, \qquad \nabla^{\mathrm{c}}\mathrm{T^c} = 0, \qquad \delta(\mathrm{T^c}) = 0.$$

We study the case of $\dim(\mathrm{G}_{\mathrm{T^c}}) \leq 2$ in a similar manner. Since $\mathrm{Hol}(\nabla^{\mathrm{c}}) \subset \mathrm{G}_{\mathrm{T^c}}$, we have the following possibilities:

$\dim(\mathrm{G}_{\mathrm{T^c}})$	0 , 1 , 2	1	2
$\dim(\mathrm{Hol}(\nabla^{\mathrm{c}}))$	0	1	1 , 2

If the holonomy group is discrete, the characteristic connection is flat and the manifold M^6 is a Lie group. Its Lie algebra is given by $\mathfrak{g} = \mathbb{R}^6$, $[X, Y] = -\mathrm{T^c}(X, Y)$. The Jacobi identity is equivalent to the condition that the square $(\mathrm{T^c})^2$ of the torsion form in the Clifford algebra $\mathfrak{Cliff}(\mathbb{R}^6)$ is a scalar (see [34, Theorem 1.50] and [37, Chapter 10.4]). However, 3-forms of type Λ^3_{12} satisfying this condition do not exist.

Lemma 4.2. *Let* $\mathrm{T} \in \Lambda^3_{12}$ *be a* 3*-form and such that its square* T^2 *in* $\mathfrak{Cliff}(\mathbb{R}^6)$ *is a scalar. Then* $\mathrm{T} = 0$.

Proof. We parameterize a form $\mathrm{T} \in \Lambda^3_{12}$ by its coefficients $A_1, \ldots, A_{12}$ with respect to the introduced basis. The endomorphism T^2 in $\mathfrak{Cliff}(\mathbb{R}^6) \subset \mathfrak{Cliff}(\mathbb{R}^7) \to \mathrm{End}(\Delta_7)$ acting in the seven-dimensional real spin representation is an (8×8)-matrix. We compute the numbers on the diagonal:

$$0, 4 \cdot (A_1^2 + A_2^2 + A_5^2 + A_6^2 + A_{11}^2 + A_{12}^2), 4 \cdot (A_3^2 + A_4^2 + A_5^2 + A_6^2 + (A_7 \pm A_{10})^2 + (A_8 \pm A_9)^2).$$

Consequently, T^2 is a scalar if and only if $\mathrm{T} = 0$. □

Theorem 4.7. *Let* (M^6, g, J) *be a complete Hermitian manifold of type* $\mathcal{W}_3$ *such that*

$$\nabla^c \mathrm{T}^c = 0, \qquad \dim(\mathrm{Hol}(\nabla^c)) = 0.$$

Then M^6 *is a flat Kähler manifold, i.e.,* $\mathrm{T}^c = 0$.

In the next step of our classification we will prove that $\dim(\mathrm{Hol}(\nabla^c)) = 1 = \dim(\mathrm{G}_{\mathrm{T}^c})$ is impossible. The holonomy representation $\mathrm{Hol}(\nabla^c) = \mathrm{G}_{\mathrm{T}^c} \to \mathrm{U}(3)$ is given by three integers $k_1, k_2, k_3 \in \mathbb{Z}$ and the diagonal matrices $\varphi \to \mathrm{diag}(\mathrm{e}^{ik_1\varphi}, \mathrm{e}^{ik_2\varphi}, \mathrm{e}^{ik_3\varphi})$. If $k_2, k_3 = 0$, the linear system $\rho_*(\omega)\mathrm{T} = 0$ has a four-dimensional solution with respect to T, namely

$$A_5 = A_6 = A_7 = A_8 = A_9 = A_{10} = A_{11} = A_{12} = 0.$$

However, a direct computation shows that for any of these 3-forms T, the stabilizer G_T is two-dimensional, i.e., both parameters $k_2, k_3 = 0$ cannot vanish. Consequently, the holonomy representation $\mathrm{G}_{\mathrm{T}^c} \to \mathrm{U}(3)$ splits into the sum of two faithful representations. The results of [14, Lemmas 4.4 and 5.6] yield again that the curvature R^c of the characteristic connection is ∇^c-parallel, $\nabla^c \mathrm{T}^c = \nabla^c \mathrm{R}^c = 0$. Since the group G_T is compact, the Ambrose–Singer manifold is regular, i.e., the manifold $M^6 = \mathrm{G}/\mathrm{G}_{\mathrm{T}^c}$ is homogeneous and the Lie algebra of its automorphism group G is the vector space $\mathfrak{g} := \mathfrak{g}_{\mathrm{T}^c} \oplus \mathbb{R}^6$ equipped with the bracket (see [14, Theorem 5.10])

$$[A + X, B + Y] = -\mathrm{R}^c(X, Y) + (A \cdot Y - B \cdot X - \mathrm{T}^c(X, Y)).$$

The curvature operator $\mathrm{R}^c : \Lambda^2(\mathbb{R}^6) \to \mathfrak{g}_{\mathrm{T}^c}$ is invariant. Fix an element $\omega \in \mathfrak{g}_{\mathrm{T}^c}$ of length one and denote by R_{ij} the coefficients of the curvature, $\mathrm{R}^c(e_i \wedge e_j) := \mathrm{R}_{ij} \cdot \omega$. Let us introduce the following element inside the Clifford algebra:

$$\mathrm{R}^c := \sum_{i<j} \mathrm{R}_{ij} \cdot e_i \cdot e_j \cdot \omega.$$

The Jacobi identity implies that the sum $(\mathrm{T}^c)^2 + \mathrm{R}^c$ is a scalar in the Clifford algebra $\mathfrak{Cliff}(\mathbb{R}^6)$ (vice versa: if $\mathrm{R} : \Lambda^2(\mathbb{R}^6) \to \mathfrak{g}_{\mathrm{T}^\mathrm{C}} \subset \Lambda^2(\mathbb{R}^6)$ is symmetric, then $(\mathrm{T}^c)^2 + \mathrm{R} \in \mathbb{R}^1$ is equivalent to the Jacobi identity). This system of equations links the curvature operator to the torsion form. We again use a suitable matrix representation of the Clifford algebra in order to discuss the system for concrete 3-forms T.

Lemma 4.3. *There is no* 5*-tuple consisting of three integers* $k_1, k_2, k_3 \in \mathbb{Z}$, *a* 3*-form* $\mathrm{T} \in \Lambda^3_{12}$ *and a curvature operator* R *such that*

(1) $\mathfrak{g}_\mathrm{T} = \mathbb{R}^1 \cdot (k_1 \cdot e_{12} + k_2 \cdot e_{34} + k_3 \cdot e_{56})$ *is one-dimensional.*

(2) *The element* $\mathrm{T}^2 + \mathrm{R}$ *is a scalar in* $\mathfrak{Cliff}(\mathbb{R}^6)$.

Proof. Since the isotropy algebra $\mathfrak{g}_\mathrm{T}$ is 1-dimensional, the 3-form is not zero. The 2-form $(k_1 \cdot e_{12} + k_2 \cdot e_{34} + k_3 \cdot e_{56})$ preserves a non-trivial element in Λ^3_{12} if and only if

$$k_1k_2k_3(k_1 + k_2 - k_3)(k_1 - k_2 + k_3)(-k_1 + k_2 + k_3) = 0.$$

Basically, there are two cases to consider: that one of the k_i's is zero, $k_3 = 0$, or that $k_3 = k_1 + k_2$.

Case 1: $k_3 = 0$. The equation $\rho_*(k_1 \cdot e_{12} + k_2 \cdot e_{34})\mathrm{T} = 0$ reads as

$$k_2A_1 = k_2A_2 = k_1A_5 = k_1A_6 = (k_1 + k_2)A_{11} = (k_1 + k_2)A_{12} = 0,$$
$$(k_1 - k_2)(A_8 \pm A_9) = (k_1 - k_2)(A_7 \pm A_{10}) = 0.$$

We split the first case into four sub-cases:

Case 1.1: $k_1 \neq 0 \neq k_2, k_1 + k_2 \neq 0, k_1 - k_2 \neq 0, k_3 = 0$. The solution space is two-dimensional and parameterized by the parameters A_3, A_4 of the 3-form $\mathrm{T} \in \Lambda^3_{12}$. Any T of that type is preserved by two elements of the Lie algebra $\mathfrak{u}(3)$, $\rho_*(e_{12})\mathrm{T} = 0 = \rho_*(e_{34})\mathrm{T}$, hence the dimension of the isotropy algebra equals two, a contradiction.

Case 1.2: $k_1 = k_3 = 0, k_2 \neq 0$. The solution space $\rho_*(k_1 \cdot e_{12} + k_2 \cdot e_{34} + k_3 \cdot e_{56})\mathrm{T} = 0$ is four-dimensional and any of these 3-forms T has a two-dimensional isotropy algebra $\mathfrak{g}_\mathrm{T}$.

Case 1.3: $k_1 \neq 0 \neq k_2, k_3 = 0, k_1 - k_2 = 0$. The solution space is six-dimensional and parameterized by the parameters A_3, A_4, A_7, A_8, A_9, A_{10}. For any of these forms, we compute the endomorphism $\mathrm{T}^2 + \mathrm{R}$ in $\mathfrak{Cliff}(\mathbb{R}^6) \subset \mathfrak{Cliff}(\mathbb{R}^7) \to \mathrm{End}(\Delta_7)$ in the seven-dimensional spin representation. The condition that $\mathrm{T}^2 + \mathrm{R}$ should be a scalar leads to the following restrictions

$$\mathrm{R}_{15} = \mathrm{R}_{16} = \mathrm{R}_{25} = \mathrm{R}_{26} = \mathrm{R}_{35} = \mathrm{R}_{36} = \mathrm{R}_{45} = \mathrm{R}_{46} = \mathrm{R}_{56} = 0,$$
$$\mathrm{R}_{23} = -\mathrm{R}_{14}, \qquad \mathrm{R}_{24} = \mathrm{R}_{13},$$
$$\mathrm{R}_{12} = -\mathrm{R}_{34} - 2\sqrt{2} \cdot (A_3^2 + A_4^2 + A_7^2 + A_8^2 + A_9^2 + A_{10}^2).$$

Moreover, the coefficients of the 3-form have to satisfy the three relations

$$A_3A_{10} = -A_4A_9, \qquad A_7A_{10} = -A_8A_9, \qquad A_3A_8 = A_4A_7.$$

The isotropy algebra $\mathfrak{g}_\mathrm{T}$ of any 3-form satisfying these conditions has dimension 2, i.e., case 1.3 is impossible.

Case 1.4: $k_1 \neq 0 \neq k_2, k_3 = 0, k_1 + k_2 = 0$. The solution space is four-dimensional and parameterized by A_3, A_4, A_{11}, A_{12}. The condition $\mathrm{T}^2 + \mathrm{R} \in \mathbb{R}^1$ for some curvature operator implies in particular that two of the parameters of the 3-form vanish, $A_{11} = A_{12} = 0$. This family of forms has been investigated already in Case 1.1. We obtain $\dim(\mathfrak{g}_\mathrm{T}) = 2$, a contradiction.

Case 2: $k_1k_2k_3 \neq 0, k_3 = k_1 + k_2$. The second case is simpler. We solve again the equation $\rho_*(k_1 \cdot e_{12} + k_2 \cdot e_{34} + (k_1 + k_2) \cdot e_{56})\mathrm{T} = 0$. The solution space is two-dimensional and parameterized by the parameters A_{11}, A_{12}. Any of these forms has a four-dimensional isotropy algebra, again a contradiction. □

A direct consequence of Lemma 4.3 is as follows.

Theorem 4.8. *Complete Hermitian manifolds* (M^6, g, J) *of type* $\mathcal{W}_3$ *such that*

$$\nabla^c \mathrm{T}^c = 0, \qquad \dim(\mathrm{G}_{\mathrm{T}^c}) = 1$$

do not exist.

Consider Hermitian $\mathcal{W}_3$-manifolds (M^6, g, J) with parallel characteristic torsion and two-dimensional isotropy group,

$$\nabla^c \mathrm{T}^c = 0, \qquad \mathrm{G}_{\mathrm{T}^c} = \mathrm{S}^1 \times \mathrm{S}^1.$$

The curvature of such a Hermitian structure is not necessarily parallel, i.e., M^6 does not have to be homogeneous. Naturally reductive Hermitian manifolds can be constructed out of a 3-form $\mathrm{T} \in \Lambda^2_{12}$ and a curvature tensor $\mathrm{R} : \Lambda^2(\mathbb{R}^6) \to \mathfrak{g}_{\mathrm{T}}$ such that the pair (T, R) defines a Lie algebra structure on $\mathfrak{g} := \mathfrak{g}_{\mathrm{T}} \oplus \mathbb{R}^6$. The naturally reductive space $\mathrm{G}/\mathrm{G}_{\mathrm{T}}$ is a Hermitian 6-manifold of type $\mathcal{W}_3$ with parallel characteristic torsion T.

Example 4.2. The isotropy algebra of the form $\mathrm{T} := e_{125} - e_{345}$ is generated by $\omega_1 := e_{12}$ and $\omega_2 := e_{34}$. The most general invariant 2-form with values in $\mathfrak{g}_{\mathrm{T}}$ depends on six parameters,

$$\mathrm{R} := \sum_{k=1}^{2} (\mathrm{R}^k_{12} \cdot e_1 \wedge e_2 + \mathrm{R}^k_{34} \cdot e_3 \wedge e_4 + \mathrm{R}^k_{56} \cdot e_5 \wedge e_6) \otimes \omega_k.$$

The Jacobi identity is equivalent to

$$\mathrm{R}^1_{56} = \mathrm{R}^2_{56} = 0, \qquad \mathrm{R}^2_{12} = \mathrm{R}^1_{34} = -1.$$

There exists a 2-parameter family of curvature operators associated with the form T,

$$\mathrm{R} = (\mathrm{R}^1_{12} \cdot e_1 \wedge e_2 - e_3 \wedge e_4) \otimes \omega_1 + (-e_1 \wedge e_2 + \mathrm{R}^2_{34} \cdot e_3 \wedge e_4) \otimes \omega_2.$$

The holonomy algebra $\mathfrak{h}$ of the connection is 1-dimensional if and only if $\mathrm{R}^1_{12} \cdot \mathrm{R}^2_{34} = 1$ holds. The Lie algebra $\mathfrak{g} = \mathfrak{g}_{\mathrm{T}} \oplus \mathbb{R}^6$ has a two-dimensional center,

$$\mathfrak{z} = \mathrm{Lin}(\omega_1 - \omega_2 + e_5, e_6).$$

Consider the Lie algebra $\mathfrak{g}^* := \mathfrak{g}/\mathfrak{z}$. Then $\mathfrak{g}$ is a central extension of $\mathfrak{g}^*$. The projections into $\mathfrak{g}^*$ of the elements $\omega_1, \omega_2, e_1, e_2, e_3, e_4$ form a basis of the vector space $\mathfrak{g}^*$ and the commutator relations in $\mathfrak{g}^*$ are given by the formulas

$$[\omega_1, \omega_2] = 0, \qquad [\omega_1, e_1] = e_2, \qquad [\omega_1, e_2] = -e_1, \qquad [\omega_1, e_3] = [\omega_1, e_4] = 0,$$
$$[\omega_2, e_1] = [\omega_2, e_2] = 0, \qquad [\omega_2, e_3] = e_4, \qquad [\omega_2, e_4] = -e_3,$$
$$[e_1, e_3] = [e_1, e_4] = [e_2, e_3] = [e_2, e_4] = 0, \qquad [e_1, e_2] = (1 - \mathrm{R}^1_{12})\omega_1,$$
$$[e_3, e_4] = (1 - \mathrm{R}^2_{34})\omega_2.$$

The Lie algebra $\mathfrak{g}^*$ is the sum of two subalgebras

$$\mathfrak{p}_1 = \mathrm{Lin}(\omega_1, e_1, e_2), \qquad \mathfrak{p}_2 = \mathrm{Lin}(\omega_2, e_3, e_4),$$

and we have

$$\mathfrak{g}^* = \mathfrak{p}_1 \oplus \mathfrak{p}_2, \qquad [\mathfrak{p}_1, \mathfrak{p}_1] \subset \mathfrak{p}_1, \qquad [\mathfrak{p}_2, \mathfrak{p}_2] \subset \mathfrak{p}_2, \qquad [\mathfrak{p}_1, \mathfrak{p}_2] = 0.$$

The Lie algebras $\mathfrak{p}_1, \mathfrak{p}_2$ are isomorphic to $\mathfrak{so}(3, \mathbb{R})$, $\mathfrak{sl}(2, \mathbb{R})$ or to the three-dimensional nilpotent Lie algebra. Consequently, we gave a complete description of the possible automorphism groups of all naturally reductive Hermitian $\mathcal{W}_3$-manifolds with parallel characteristic torsion of type $\mathrm{T} = e_{125} - e_{345}$.

We remark that the torsion form $\mathrm{T} = e_{125} - e_{345}$ represents the general case. Indeed, let $\mathrm{T} \in \Lambda^3_{12}$ be a 3-form with a two-dimensional isotropy group. Following once again carefully the proof of Lemma 4.3, we see that this form behaves like $e_{125} - e_{345}$ in the sense that the automorphism groups are the same. Therefore, we obtain the following theorem.

Theorem 4.9. *Any naturally reductive Hermitian $\mathcal{W}_3$-manifold with a two-dimensional isotropy algebra $\mathfrak{g}_{\mathrm{T}^c}$ of its characteristic torsion is locally isometric to one of the spaces described in the previous example.*

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